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# ON THE DEFINABILITY OF RADICALS IN SUPERSIMPLE GROUPS

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**Abstract.** If $G$ is a group with a supersimple theory having a finite $SU$-rank, then the subgroup of $G$ generated by all of its normal nilpotent subgroups is definable and nilpotent. This answers a question asked by Elwes, Jaligot, Macpherson and Ryten. If $H$ is any group with a supersimple theory, then the subgroup of $H$ generated by all of its normal soluble subgroups is definable and soluble.

**§1. Introduction.** Among the problems in the model theory of groups, is the one of knowing which subgroups of a group $G$ are definable by a formula. For example, the centraliser of an element $a$ in $G$ is defined by the quantifier free formula $xa = ax$ and the centre of $G$ by $(\forall y)\ xy = yx$. Similarly, finite sets, centralisers of finite sets and iterated centres of $G$ are always definable. But this is mostly the end of the list: almost every other characteristic subgroup such as the commutator subgroup $G'$, the $FC$-centre, the Fitting subgroup or the soluble radical may not be definable, not in first order logic at least: they all are countable union of definable sets. The situation is even more complicated for the iterated $FC$-centres, the $FC$-soluble radical or the $FC$-Fitting subgroup who have a higher complexity in the hierarchy of definable sets.

In an algebraic group over an algebraically closed field, every subgroup cited above is definable. The situation is far less straightforward in a group $G$ which is merely stable. Wagner has shown that the Fitting subgroup of $G$ is always definable [18]. The question is still open for the soluble radical of $G$, but Baudish [3] has proved that it is definable provided that $G$ be superstable. The starting point of their investigation was a theorem of Poizat [14] that every nilpotent (respectively soluble) subgroup of $G$ is contained in a definable nilpotent (respectively soluble) one of the same nilpotency class (resp. derived length). Recently, many attempts have been made to extend these results to a wider context: let us cite [17, Shelah] and [1, Aldama] for groups with dependent theory, [2, Altınel Baginski] for groups with the descending chain condition on centralisers, [12, Milliet] for groups with a simple theory and [6, Elwes Jaligot Macpherson Ryten], for supersimple groups, where it is shown that the soluble radical of a supersimple group $G$ of finite rank is definable

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and soluble provided that $G^{eq}$ eliminates $\exists^\infty$. Note that the formulation *supersimple group* is a short-hand for *group with a supersimple theory*, not a variation on *simple groups*. The authors of [6] also asked whever such a group $G$ had a largest nilpotent normal subgroup $F$ and if $F$ would be definable. We give a positive answer here while proving that the Fitting subgroup of a supersimple group of finite $SU$-rank is definable and nilpotent. We also show that the soluble radical of a supersimple group of arbitrary rank is a definable and soluble subgroup. As a corollary, the $FC$-soluble radical of a supersimple group is virtually soluble and definable.

§**2. Preliminaries on groups.** If $G$ is a group and $x$ an element of $G$, we write $x^G$ for the conjugacy class $\{g^{-1}xg : g \in G\}$ of $x$, and $C(x)$ for its centraliser $\{g \in G : g^{-1}xg = x\}$ in $G$. If $y$ is another element of $G$, we write $[x, y]$ for the commutator $x^{-1}y^{-1}xy$. When $A$ and $B$ are subsets of $G$, we write $[A, B]$ for the set of commutators $[a, b]$ where $a$ and $b$ range over $A$ and $B$ respectively. We write $G^{(n)}$ for the $n$th term of the derived series of $G$ defined inductively on $n$ by putting $G^{(0)}$ equal to $G$ and $G^{(n+1)}$ the subgroup generated by the set $[G^{(n)}, G^{(n)}]$. The group $G$ is *soluble of derived length n* if $n$ is the smallest natural number such that $G^{(n)}$ is $\{1\}$.

The *FC-centre* of a group $G$ is written $FC(G)$ and is defined to be the set of $g$ in $G$ such that $g^G$ is finite. By definition, the group $G$ is an *FC-group* if $FC(G)$ equals $G$. Inductively on $n$, we call $FC_{n+1}(G)$ the preimage in $G$ of $FC(G/FC_n(G))$, with the convention that $FC_0(G)$ is $\{1\}$. This defines an ascending chain of characteristic subgroups of $G$. The group $G$ is called *FC-nilpotent* if $G$ equals $FC_n(G)$ for some natural number $n$, the least such we call the *FC-nilpotency class of G*. Finite groups and nilpotent ones are both examples of $FC$-nilpotent groups. If $G/N$ is a quotient group modulo a normal subgroup $N$ of $G$, we write $FC_G(G/N)$ for the preimage of $FC(G/N)$ in $G$ by the canonical surjection from $G$ onto $G/N$.

A group $G$ is *virtually-P* if it has a subgroup of finite index with property $P$.

Theorem 2.1 (Neumann [13]). *Suppose that $G$ is an $FC$-group whose conjugacy classes are bounded by a natural number. Then the derived subgroup $G'$ is finite and $G$ is virtually nilpotent of class* 2.

Lemma 2.2. *If $N$ is a finite normal subgroup of $G$, then $FC(G)$ equals $FC_G(G/N)$.*

Proof. The canonical surjection from $G$ onto $G/N$ has a finite kernel. It follows that the conjugacy class $x^G$ is finite if and only if $(xN)^{G/N}$ is finite. ⊣

Lemma 2.3. *If $H$ and $N$ are two normal subgroups of $G$ with $N \leq H$, then $FC_G(G/H)/N$ equals $FC_{G/N}\big(G/N\big/H/N\big)$.*

Proof. There is a canonical homomorphism from $G/H$ onto $G/N\big/H/N$. It follows that $(xH)^{G/H}$ is finite if and only if $\big((xN)H/N\big)^{G/N\big/H/N}$ is finite. This means precisely that $x \in FC_G(G/H)$ if and only if $xN \in FC_{G/N}\big(G/N\big/H/N\big)$. ⊣

Lemma 2.4. *If for some natural number $n$ the quotient $FC_{n+1}(G)/FC_n(G)$ is finite, then $FC_{n+2}(G)$ equals $FC_{n+1}(G)$.*

Proof. We have

$$\frac{FC_{n+2}(G)}{FC_n(G)} = \frac{FC_G\big(G/FC_{n+1}(G)\big)}{FC_n(G)}.$$

By Lemma 2.3

$$\frac{FC_{n+2}(G)}{FC_n(G)} = FC_{G/FC_nG}\left(\frac{G/FC_n(G)}{FC_{n+1}(G)/FC_n(G)}\right).$$

As $FC_{n+1}(G)/FC_n(G)$ is finite, applying Lemma 2.2 we get

$$\frac{FC_{n+2}(G)}{FC_n(G)} = FC\left(\frac{G}{FC_n(G)}\right) = \frac{FC_{n+1}(G)}{FC_n(G)}.$$ ⊣

Two subgroups of a given group $G$ are *commensurable* if the index of their intersection is finite in both of them. Commensurability is an equivalence relation on the set of subgroups of $G$.

THEOREM 2.5 (Schlichting [16]). *Let $G$ be a group and $H$ a subgroup of $G$ such that $H/H \cap H^g$ remains finite and bounded by a natural number for all $g$ in $G$. Then, there exists a normal subgroup $N$ of $G$ such that $H/H \cap N$ and $N/N \cap H$ are finite. Moreover, $N$ is a finite extension of a finite intersection of $G$-conjugates of $H$. In particular, if $H$ is definable then so is $N$.*

§3. **Preliminaries on supersimple groups.** A supersimple group $G$ is equipped with a rank function taking values in the ordinals, and ranking every definable subset of $G$. We write $SU(X)$ for the rank of a definable subset $X$ of $G$. As there is no other rank considered in the paper, we will simply say *rank* instead of *$SU$-rank*. We shall not need the precise definition of the rank (we refer to [19] for more details), but only some of its properties that we recall now. The rank is increasing: if $X \subset Y$ are two definable subsets of $G$, then $SU(X)$ is smaller than or equal to $SU(Y)$. If $G$ is supersimple, then so is each of its elementary extensions, and so is $G^{eq}$, meaning that every quotient group $G/N$ by a definable normal subgroup $N$ has an ordinal rank. A definable set (in $G^{eq}$) has rank zero if and only if it is finite. In particular, if $N$ is a definable normal subgroup of $G$, then $SU(G/N)$ equals zero if and only if $N$ has finite index in $G$.

The following comes from [7, Remark 3.5] as a particular case of [19, Theorem 5.5.4].

THEOREM 3.1 (Wagner's version of Zilber's Indecomposability Theorem). *Let $G$ be a supersimple group of finite rank, $(X_i)_{i \in I}$ a family of definable subsets of $G$. Then, there exists a definable subgroup $H$ of $G$ such that*

(1) *$H$ is a subgroup of $\langle X_i : i \in I \rangle$.*
(2) *Finitely many translates of $H$ cover $X_i$ for every $i$.*

*If the sets $X_i$ are normal in $G$, then $H$ may be chosen normal in $G$.*

COROLLARY 3.2. *If $G$ is a supersimple group with finite rank, then the derived subgroup $G'$ is definable.*

PROOF. We follow exactly the proof of [9, Corollary 7.5]. Let $C$ be the set of commutators of $G$. By Theorem 3.1, there is a definable subgroup $H$ of $G'$ with $H$ normal in $G$ such that finitely many translates of $H$ cover $C$. It follows that the set of commutators in $G/H$ is finite, so the derived group $(G/H)'$ is finite by [10, p. 110]. The group $G'$ is a finite union of cosets of $H$ hence definable. ⊣

Any ordinal $\alpha$ decomposes in base $\omega$: there are unique ordinals $\alpha_1 > \cdots > \alpha_n$ and non-zero natural numbers $k_1, \ldots, k_n$ such that $\alpha$ equals $\omega^{\alpha_1}.k_1 + \cdots + \omega^{\alpha_n}.k_n$.

If $\alpha$ and $\beta$ are two ordinals, we may assume that $\alpha$ equals $\omega^{\alpha_1}.k_1 + \cdots + \omega^{\alpha_n}.k_n$ and $\beta$ equals $\omega^{\alpha_1}.\ell_1 + \cdots + \omega^{\alpha_n}.\ell_n$ for the same $\alpha_1, \dots, \alpha_n$, adding some additional possibly zero $k_i$ and $\ell_i$ if necessary. We write $\alpha \oplus \beta$ for their *Cantor sum* defined by

$$\alpha \oplus \beta = \omega^{\alpha_1}.(k_1 + \ell_1) + \cdots + \omega^{\alpha_n}.(k_n + \ell_n).$$

THEOREM 3.3 (Lascar inequalities). *Let $G$ be a supersimple group, and $H$ a definable normal subgroup of $G$. Then*

$$SU(H) + SU(G/H) \leq SU(G) \leq SU(H) \oplus SU(G/H).$$

As a consequence, note that two definable subgroups of a supersimple group which are commensurable have the same rank.

PROPOSITION 3.4. *Let $G$ be a supersimple group of rank $\omega^{\alpha_1}.k_1 + \cdots + \omega^{\alpha_n}.k_n$ with $\alpha_1 > \cdots > \alpha_n$. Then for every natural number $i$ such that $1 \leq i \leq n$, there is a definable normal subgroup $H$ of $G$ of rank $\omega^{\alpha_1}.k_1 + \cdots + \omega^{\alpha_i}.k_i$. The group $H$ is unique up to commensurability.*

*Remark* 3.5. Proposition 3.4 is the definable version of [20, Wagner, Corollary 4.2]. It generalises what is known for superstable groups [4, Corollary 2.7 p. 27].

PROOF. We may assume that $G$ is $\kappa$-saturated for some infinite cardinal $\kappa$ and we say that a set is *small* if its cardinal is smaller than $\kappa$. We write $\beta_i$ for $\omega^{\alpha_1}.k_1 + \cdots + \omega^{\alpha_i}.k_i$. By [20, Corollary 4.2], there is a type-definable normal subgroup $H$ of $G$ having rank $\beta_i$. Recall that $\beta_i$ is by definition the rank of each of the generic types of $H$. By [20, Theorem 4.4], the group $H$ is the intersection of definable groups $H_i$ for $i$ in $I$. We may close this family by finite intersections, remove the members that do not have minimal rank and assume that every $H_i$ has rank $\beta$ say and that they are all commensurable. It follows that for every $i$, the group $H$ has small index in $H_i$ so $H$ is a generic type of $H_i$ by [19, Lemma 4.1.15]. Thus $\beta$ equals $\beta_i$. Take any $H_i$. As $H$ is normal in $G$, $H_i^g$ and $H_i$ are commensurable for every $g$ in $G$. Let $FN(H_i)$ stand for the set of $g$ in $G$ such that $H_i/H_i \cap H_i^g$ is finite. On the one hand, the group $FN(H_i)$ is the countable union of the definable sets $FN_m(H_i)$ when $m$ ranges over $\mathbf{N}$ and where $FN_m(H_i)$ stands for $\{g \in G \colon |H_i \colon H_i \cap H_i^g| \leq m\}$. On the other hand, by [19, Lemma 4.1.15] and [19, Remark 4.1.5], it is type-definable. It must be definable by compactness and saturation. It follows that $H_i/H_i \cap H_i^g$ remains bounded by some natural number when $g$ ranges over $G$. By Theorem 2.5, there is a definable normal subgroup $N$ of $G$ commensurable with $H$ hence of rank $\beta_i$. If $K$ is another group satisfying the desired requirements, then $K/N \cap K$ and $N/N \cap K$ are small according to [20, Corollary 4.2] hence finite by compactness and saturation. ⊣

LEMMA 3.6. [12, Proposition 4.1] *If $G$ is a group with (super)simple theory, its $FC$-centre is definable.*

PROOF. It is shown in [12] that $FC(\mathbf{G})$ is definable by a formula $\psi$ provided that $\mathbf{G}$ is an $\aleph_0$-saturated extension of $G$. Actually the same formula $\psi$ computed in $G$ defines $FC(G)$. ⊣

§**4. The Fitting subgroup.** Let $G$ be any group. We call the *Fitting subgroup of* $G$ the subgroup generated by all of its nilpotent normal subgroups. We write it as $Fit(G)$. It is worth mentioning that the Fitting subgroup is definable if it is nilpotent. Namely $x$ belongs to $Fit(G)$ if and only if the subgroup generated by its conjugacy class $x^G$ is nilpotent. It follows that

$$Fit(G) = \bigcup_{n \geq 1} \left\{ x \in G \colon \left[ x^G \cup x^{-G}, \ldots, {}_n x^G \cup x^{-G} \right] = \{1\} \right\}$$

where for every subset $X$ of $G$ the set $[X, \ldots, {}_n X]$ is defined inductively on $n \geq 0$ by

$$[X, {}_0 X] = X \quad \text{and} \quad [X, \ldots, {}_{n+1} X] = \big[ [X, \ldots, {}_n X], X \big].$$

The observation that $Fit(G)$ is definable if it is nilpotent was first made by Ould Houcine, and the simple proof above was independently provided by the referee of [2].

PROPOSITION 4.1. *Let $G$ be a group and $F$ a normal subgroup of $G$. Assume that $F \leq FC_n(G)$ for some natural number $n$ (in particular, $F$ is $FC$-nilpotent). If $G/F$ is $FC$-nilpotent, then so is $G$.*

PROOF. Assume that $G/F$ is $FC$-nilpotent of class $m$. There is a surjection from $G/F$ onto $G/FC_n(G)$. As recalled in [12], the image of an $FC$-nilpotent group by a group homomorphism is $FC$-nilpotent. It follows that $G/FC_n(G)$ is $FC$-nilpotent of class at most $m$, so that $FC_{m+n}(G)$ equals $G$. ⊣

We recall Hall's criterion for nilpotence.

THEOREM 4.2 (Hall [8]). *Let $G$ be a group and $N$ a normal subgroup of $G$. If $G/N'$ and $N$ are nilpotent, then $G$ is nilpotent.*

Two other proofs of Theorem 4.2 can be found in [15] and [11], with a bound on the nilpotency class of $G$ depending on the classes of $N$ and $G/N'$ in [11]. Note that since $G/N''/N'/N''$ and $G/N'$ are isomorphic, a straightforward induction on the nilpotency class of $N$ reduces the proof to the case where $N$ is 2-nilpotent.

PROPOSITION 4.3 (adapted from Wagner [12, Proposition 4.3]). *Let $G$ be a group with a (super)simple theory. If $G$ is $FC$-nilpotent of class $n$, then $G$ has a definable normal subgroup of finite index which is nilpotent of class at most $2n$.*

THEOREM 4.4 (Milliet [12, Corollary 4.5]). *Let $G$ be a group with a (super)simple theory. If $N$ is a normal nilpotent subgroup of class $n$, then $N$ is contained in a normal definable nilpotent subgroup of class at most $3n$.*

We can now answer the question asked in [6].

THEOREM 4.5. *Let $G$ be a supersimple group with finite rank. The Fitting subgroup of $G$ is definable and nilpotent.*

PROOF. By Lemma 2.4, for big enough $n$, the quotient $FC_{n+1}(G)/FC_n(G)$ is either trivial or infinite. By Lascar's equality, there exists a natural number such that $FC_n(G) = FC_{n+1}(G)$. We call $G_n$ the quotient group $G/FC_n(G)$ so that $G_n$ has a trivial $FC$-centre. Let $F_n$ be its Fitting subgroup.

CLAIM 1. *We may assume that $FC(G)$ is trivial.*

PROOF OF CLAIM 1. We need just assume that $F_n$ is definable and nilpotent and show that $Fit(G)$ is definable and nilpotent too. Note that $F_n$ is $FC$-nilpotent. Let $F$ be its pull-back in $G$ so that we have $F/FC_n(G) = F_n$. The group $F$ is $FC$-nilpotent by Proposition 4.1, and definable. By Proposition 4.3, $F$ has a definable subgroup of finite index which is nilpotent, so it must have a normal one $N$. It follows that $F$ contains a maximal normal (in $G$) nilpotent subgroup $H$ of finite index so that $H$ equals $Fit(G)$. Being a finite extension of $N$, $Fit(G)$ is definable. $\dashv_{\text{Claim 1}}$

Claim 2. *We may assume that $G$ has a definable normal 2-nilpotent subgroup.*

proof of claim 2. On the one hand, if every nilpotent normal subgroup of $G$ is abelian, then $Fit(G)$ is abelian. In this case, by the remark made at the beginning of this section (or Theorem 4.4), $Fit(G)$ must be definable. On the other hand, if there is a non abelian nilpotent normal subgroup, then there is a definable one by Theorem 4.4. Call it $N$. The group $Z_2(N)$ has the required properties. $\dashv_{\text{Claim 2}}$

We proceed by induction on $SU(G)$ to prove Theorem 4.5. If $SU(G)$ is zero, then $G$ is finite and so is $Fit(G)$. If $SU(G)$ equals $n+1$, by Claim 2, there is a normal nilpotent definable subgroup $N$ of $G$ of nilpotency class 2. By Corollary 3.2, the derived subgroup $N'$ is definable. As $N'$ is normal in $G$, it is infinite by Claim 1, so $SU(N') \geq 1$. By Lascar's equality, we have $SU(G/N') \leq n$, and we may apply the induction hypothesis to $G/N'$. It follows that $Fit(G/N')$ is definable and nilpotent. Let $F$ be its preimage in $G$ so that we have $F/N' = Fit(G/N')$. By Theorem 4.2, the group $F$ is nilpotent. Thus $F$ equals $Fit(G)$. $\dashv$

**§5. The soluble radical.** We call the *soluble radical of $G$* the subgroup generated by all soluble normal subgroups and write it as $R(G)$. It is a locally soluble subgroup.

We recall a simple and useful remark by Ould Houcine:

Lemma 5.1 (Ould Houcine). *Let $G$ be any group and suppose that $R(G)$ is soluble. Then $R(G)$ is definable.*

Proof. An element $x$ belongs to $R(G)$ if and only if the subgroup generated by its conjugacy class $x^G$ is soluble. Note that the derived subgroup $\langle x^G\rangle^{(1)}$ is generated by all commutators of the form $[a^g, b^h]$ where $a$ and $b$ equal $x$ or $x^{-1}$ and $g$ and $h$ range over $G$. Thus the following equality holds

$$R(G) = \bigcup_{n\geq 1} \left\{ x \in G \colon \left(x^G \cup x^{-G}\right)^{(n)} = \{1\} \right\}$$

where for every subset $X$ of $G$ the set $X^{(n)}$ is defined inductively on $n \geq 0$ by

$$X^{(0)} = X, \quad X^{(1)} = [X, X] \quad \text{and} \quad X^{(n+1)} = \left[X^{(n)}, X^{(n)}\right].$$

It follows that $R(G)$ is a countable union of increasing definable sets. As $R(G)$ is soluble, this union is actually a finite one, and $R(G)$ is definable. $\dashv$

Theorem 5.2 (Milliet [12, Corollary 4.11]). *Let $G$ be a group with a (super)simple theory and $S$ a normal soluble subgroup of derived length $n$. Then $S$ is contained in a definable soluble subgroup of derived length at most $3n$.*

Theorem 5.3. *The soluble radical of a supersimple group is definable and soluble.*

Proof. We shall proceed by transfinite induction on the rank of $G$. For that, we first show the following claim:

Claim. *If $G$ has a normal subgroup $H$ such that both $R(G/H)$ and $R(H)$ are soluble, then $R(G)$ is definable and soluble.*

Proof of the claim. Let $\ell$ be a natural number bounding the derived length of both $R(G/H)$ and $R(H)$. Let $S$ be a normal soluble subgroup of $G$ of derived length $n$. It follows that $SH/H$ and $S \cap H$ are soluble of derived length no greater than $\ell$. As $SH/H$ is isomorphic to $S/S \cap H$, we must have $S^{(\ell)} \subset S \cap H$ which in turn yields $S^{(2\ell)} = \{1\}$. It follows that $n$ is less than or equal to $2\ell$, so that the solubility class of $S$ is bounded, independently on $S$. This means that $R(G)$ is soluble, hence definable by Lemma 5.1. $\dashv_{\text{Claim}}$

We can now inductively prove our theorem: if $G$ has rank zero, then $G$ is finite and so is $R(G)$.

If $G$ has a non-monomial rank, there is a natural number $k > 0$ and ordinals $\alpha$ and $\beta$ such that $SU(G)$ equals $\omega^\alpha.k + \beta$ with $0 < \beta < \omega^\alpha$. By Proposition 3.4, there is a normal subgroup $H$ of $G$ having rank $\omega^\alpha.k$. By the Lascar inequalities, both $SU(G/H)$ and $SU(H)$ are less than $SU(G)$ so we may apply the induction hypothesis to $H$ and $G/H$ and it follows from the Claim that $R(G)$ is soluble and definable.

If $G$ has a monomial rank, it is of the form $\omega^\alpha.k$. Let us first suppose that there is some $a$ in $R(G)$ with $a^G$ having rank at least $\omega^\alpha$. By Theorem 5.2, the conjugacy class $a^G$ is contained in a definable normal soluble group $S$. As the rank is increasing, $S$ must have rank at least $\omega^\alpha$. In that case, either $S$ and $G$ have the same rank so $G$ is virtually soluble and we are done, or $SU(S) < SU(G)$. Then, by the Lascar inequalities 3.3 we have $SU(G/S) < SU(G)$ and we may again apply the Claim. One last case to deal with: we may have $SU(a^G) < \omega^\alpha$ for all $a$ in $R(G)$. As $a^G$ and $G/C(a)$ are in definable bijection it follows that $SU(G/C(a)) < \omega^\alpha$ for all $a$ in $R(G)$. By the Lascar inequalities, this is equivalent to saying that $SU(G/C(a))$ is zero for all $a$ in $R(G)$. So $R(G)$ is a subgroup of the $FC$-centre of $G$ which is definable by Lemma 3.6 and virtually nilpotent of class 2 by Theorem 2.1. It follows that $R(G)$ is also virtually nilpotent of class 2 (and locally soluble) hence soluble. In every case, $R(G)$ is definable and soluble. $\dashv$

## §6. The $FC$-soluble radical.

Definition 6.1 (adapted from Duguid, McLain [5]). A group $G$ is *$FC$-soluble* if there exists a finite sequence of subgroups $G_0, G_1, \dots, G_n$ of $G$ such that

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$$

and such that $G_i/G_{i+1}$ is an $FC$-group for all $i$. We call the least such natural number $n$ the *$FC$-solubility class of $G$*.

If $N$ is a normal subgroup of $G$, then $G$ is $FC$-soluble if and only if $G/N$ and $N$ are $FC$-soluble.

We define the *$FC$-soluble radical* of a group to be the subgroup generated by every normal $FC$-soluble subgroup. This is a locally $FC$-soluble subgroup:

Lemma 6.2. *Let $H$ and $K$ be two normal $FC$-soluble subgroups of a group $G$ of class $h$ and $k$. The product $HK$ is $FC$-soluble of $FC$-solubility class at most $h + k$.*

Proof. The quotient $HK/K$ is isomorphic to $H/H \cap K$. So $K$ and $HK/K$ both are $FC$-soluble. $\dashv$

Proposition 6.3 (Milliet [12, Corollary 4.9]). *A (super)simple $FC$-soluble group is virtually-soluble.*

COROLLARY 6.4. *The FC-soluble radical of a supersimple group is definable and virtually soluble.*

PROOF. By Theorem 5.3, $R(G)$ is definable so the quotient $G/R(G)$ is supersimple and has no non-trivial normal soluble subgroup. Let us write it as $G_R$. By Proposition 6.3, an $FC$-soluble subgroup of $G_R$ is virtually-soluble, hence finite, so every normal $FC$-soluble subgroup is contained in $FC(G_R)$. By Lemma 3.6 and Theorem 2.1, the group $FC(G_R)$ must be finite. Its preimage in $G$ is definable, virtually soluble, and contains every normal $FC$-soluble subgroup of $G$. ⊣

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