Model theory
1. Structures, morphisms

Exercise 1 (on substructures) Let $L_{gp} = \{\times, ^{-1}, 1\}$ be the language of groups and let G be a group considered as an L_{gp}-structure where $\times$, $^{-1}$ and 1 have their natural interpretations.

1. Show that an L_{gp}-substructure of G is a subgroup of G, and reciprocally that a subgroup of G is an L_{gp}-substructure of G. In the reduced language $L_s = \{^{-1}\}$ interpreted naturally in G, how can you describe the L_s-substructures of G?

2. Let L be any language, M an L-structure, and $\{M_i : i \in I\}$ a family of L-substructures of M. Show that the intersection $\bigcap_{i \in I} M_i$ is again an L-substructure of M when it is non-empty.

3. Recall that for a subset A of G, the *subgroup generated by* A is the intersection of all the subgroups of G containing A. Show that the subgroup generated by A is precisely the L_{gp}-substructure generated by A. What is the L_s-substructure of G generated by A?

4. If L is any language, M an L-structure and $B \subset M$, show that the domain of $\langle B \rangle$ is the smallest subset of M containing B, the constants of L^M and closed under the functions of L^M.

Exercise 2 (on the isomorphism relation) Let L be a language, (M, L^M), (N, L^N), (S, L^S) three L-structures.

1. An example first: if L_{ring} is the language of rings, R_1 and R_2 two rings with their natural L_{ring}-structures. Show that σ is a ring isomorphism from R_1 to R_2 if and only if σ is an L_{ring}-isomorphism from R_1 to R_2.

2. Back to the general case. Show that if α is an L-isomorphism from M to N, and β an L-isomorphism from N to S, then $\beta \circ \alpha$ is an L-isomorphism from M to S.

3. If α is an L-isomorphism from M to N, show that α^{-1} is an L-isomorphism from N to M. Show that the relation 'there is an L-isomorphism from M to N' defines an equivalence relation on the class of all L-structures.

Exercise 3 (on the automorphism groups) Let L be a language, (M, L^M) an L-structure, A a subset of M. Let us write $Aut(M)$ for the set of all L-automorphisms of M, $Aut(M/\{A\})$ for the set consisting of σ in $Aut(M)$ that fix A setwise (*i.e.* that satisfy $\sigma(a) \in A$ and $\sigma^{-1}(a) \in A$ for all a in A), and $Aut(M/A)$ for the σ in $Aut(M)$ that fix A pointwise (*i.e.* that satisfy $\sigma(a) = a$ for all a in A).

1. Show that $Aut(M)$ is a group (for the composition law), that $Aut(M/\{A\})$ is a subgroup of $Aut(M)$ and that $Aut(M/A)$ is a normal subgroup of $Aut(M/\{A\})$. Is $\{\sigma \in Aut(M) : \sigma(A) \subset A\}$ always a subgroup of $Aut(M)$? Show that $\{x \in M : \sigma(x) = x\}$ is an L-substructure of M for all σ in $Aut(M)$.

2. An example. Let L_{field} be the language of fields and let K be a field with its natural L_{field} structure (to interpret $^{-1}$ in 0, we adopt the convention that $0^{-1} = 0$). If A is a subset of K, show that $Aut(K/A)$ is the *Galois group* $\mathrm{Gal}(K/F)$ of a field extension K/F to be determined (we define $\mathrm{Gal}(K/F)$ as the group of field automorphisms of K fixing the subfield F pointwise).

Exercise 4 (a bijective L-morphism need not be an L-isomorphism) Give an example of a language L, two L-structures M and N and a bijective L-morphism from M to N which is not an L-isomorphism.