Model theory
10. Quantifier elimination

Exercise 1 (projections of semi-algebraic sets) For any set X and natural numbers $n \geqslant 1$, $m \geqslant 0$ the map $\pi : X^{n+m} \longrightarrow X^n$, $(x_1, \ldots, x_{n+m}) \mapsto (x_1, \ldots, x_n)$ is called a *projection*. Let M be an L-structure and Σ its theory. A subset A of M^n is *definable* if there is a formula $\varphi(\bar{x})$ such that $A = \{\bar{a} \in M^n : M \models \varphi(\bar{a})\}$.

1. Show that Σ has quantifier elimination if and only if every L-sentence is equivalent modulo Σ to a quantifier-free one and, for every $n \geqslant 1$, $m \geqslant 0$ and every definable subset $A \subset M^{n+m}$ defined by a quantifier-free formula, $\pi(A)$ is also definable by a quantifier-free formula.

2. A *semi-algebraic* set is a subset of $\mathbf{R}^n$ defined by a finite Boolean combination of polynomial equations $p_i(x_1, \ldots, x_n) = 0$ and polynomial inequalities $q_j(x_1, \ldots, x_n) > 0$ where p_i and q_j have real coefficients. Show that the projection of a semi-algebraic set is a semi-algebraic set.

Exercise 2 Let Σ be an L-theory. Show that Σ has quantifier elmination if and only if for all models M and N of Σ and all L-structure A with $A \subset_L M$ and $A \subset_L N$, one has $M \equiv_A N$.

Exercise 3

1. What are the subsets of $\mathbf{N}$ definable by a quantifier-free formula in the language $(0, <)$? Does $\mathbf{N}$ have quantifier elimination in that language?

2. What are the subsets of $\mathbf{Z}$ definable by a quantifier-free formula in the language $(0, 1, +, -, <)$? Does $\mathbf{Z}$ have quantifier elimination in that language?

3. Does $\mathbf{R}$ have quantifier elimination in the language of rings?

4. Does the theory of Abelian groups has quantifier elimination in the language $(+, 0)$?