Model theory
11. Categoricity

Exercise 1 (algebraically closed fields)

1. Let K be an infinite field and F a subfield. Show that if K/F is algebraic and F infinite, then K and F have the same cardinality. Let B be a transcendence basis of K over F. Express $|K|$ in terms of $|B|$ and $|F|$.

2. Show that the L_{ring}-theory of an algebraically closed field is not $\aleph_0$-categorical.

3. Show that the L_{ring}-theory of an algebraically closed field is λ-categorical for every $\lambda > \aleph_0$.

Exercise 2 (K-vector spaces) Let K be an infinite field and L_K the language $\{+, m_k, 0 : k \in K\}$ where m_k is unary function symbol for every k. A K-vector space V has a natural L_K-structure where m_k is interpreted putting $m_k^V(x) = kx$ for every x in V. Let Σ be the theory of all vector-spaces.

1. Write down the axioms for Σ.

2. Show that Σ is not $|K|$-categorical.

3. Show that Σ is λ-categorical for every $\lambda > |K|$. Is Σ complete?

4. Does Σ eliminate quantifiers?

Exercise 3 Let L be the language $\{=, c_n : n \in \mathbf{N}\}$ where c_n is a constant symbol for every n. Let $\mathbf{N}$ be the L-structure such that $c_n^{\mathbf{N}} = n$, and let Σ be its theory. Let λ be an infinite cardinal. Is Σ λ-categorical? Does Σ eliminate quantifiers?