

Model theory

12. Definability

Exercise 1 (number of definable sets) Let M be an L -structure. Compare the cardinality of the set of definable subsets of M with the set of all subsets of M . Give a few examples.

Exercise 2 (definable maps) Let M be an L -structure and $f : M^n \rightarrow M^m$ a map. We say that f is definable if its graph is a definable subset of M^{m+n} .

1. Let $f : M^n \rightarrow M^m$ and $g : M^m \rightarrow M^\ell$. If f and g are definable, is $g \circ f$ definable? If $g \circ f$ is definable, are f and g definable?
2. Let $f : M^n \rightarrow M$. If f is definable, is the image of f definable? If the image of f is definable, is f definable?
3. If $f : M^n \rightarrow M$ is a definable bijection, is f^{-1} definable?

Exercise 3 (on interpretability) 1. Let L_1, L_2, L_3 be three languages, M_i an L_i -structure for each $i \in \{1, 2, 3\}$. M_3 is interpretable in M_2 and M_2 is interpretable in M_1 , show that M_3 is interpretable in M_1 .

2. Show that if M is an infinite L -structure, then any finite structure (in a finite language) is interpretable in M .
3. Let K be an infinite field, and L/K be a finite algebraic extension. Show that L is interpretable in K . Is K interpretable in L ?

Exercise 4 (one types in \mathbf{Q}) Determine all the 1-types over \mathbf{Q} in the ordered set $(\mathbf{Q}, <)$. What is the cardinality of $S_1(\mathbf{Q})$?