Model theory - Sheet 2

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Exercise 1

Claim. In $L := L_{ring}$, for the natural ring structure on $\mathbb{R}$, the functions $t^{\mathbb{R}}$ for L-terms t are precisely the polynomial functions on $\mathbb{R}$ having coefficients in $\mathbb{Z}$. The smallest L-substructure of $\mathbb{R}$ ist $\mathbb{Z}$.

Proof. Let t be an L-term. We show by induction on $c(t)$ that $t^{\mathbb{R}}$ ist a polynomial function with coefficients in $\mathbb{Z}$.

If t is a constant, then $t^{\mathbb{R}} = 0$ or $t^{\mathbb{R}} = 1$, which are polynomial functions with coefficients in $\mathbb{Z}$. If t is the variable x_i, then $t^{\mathbb{R}}$ is the monomial X_i.

If t is of the form $f(t_1, ..., t_n)$ with $t_1, ..., t_n$ L-terms, then $f^{\mathbb{R}} = +$ or $f^{\mathbb{R}} = \times$. In the first case, by induction hypothesis, $t^{\mathbb{R}}$ is the sum $t_1^{\mathbb{R}} + t_2^{\mathbb{R}}$ of two polynomial functions with coefficients in $\mathbb{Z}$, so it is such a function. In the second case, $t^{\mathbb{R}}$ ist the product of two such functions, therefore it is itself such a function.

For the converse, we first note that for any constant $a \in \mathbb{Z}$, we have that the function $x \mapsto a$ is given by $t^{\mathbb{R}}$ with $t = 0 \pm (1 + ... + 1)$, where we add up $|a|$ ones and choose a suitable sign. Furthermore, $x \mapsto x^n$ is given by $t^{\mathbb{R}} = x \cdot ... \cdot x$, where we multiply $n-1$ times. Inductively, we find that any monomial of the form $a x_1^{n_1} \cdot ... \cdot x_r^{n_r}$ with $a \in \mathbb{Z}$ is given as $t^{\mathbb{R}}$ for some L-term t and, again by induction, any finite sum of such monomials is given as $t^{\mathbb{R}}$ for some term t. But these are just all the polynomial functions on $\mathbb{R}$ having coefficients in $\mathbb{Z}$.

Let (S, L^S) be any L-substructure of $\mathbb{R}$. Then, $0 = 0^{\mathbb{R}}$ and $1 = 1^{\mathbb{R}}$ are in S. Since S is closed under addition, we inductively find that $\mathbb{N} \subseteq S$. Since $0 \in S$ and S is closed under subtraction, we also find that $-\mathbb{N} \subseteq S$. Therefore, $\mathbb{Z} \subseteq S$. Conversely, $\mathbb{Z}$ is a subring of $\mathbb{R}$ containing 0 and 1 so it indeed is an L-substructure of $\mathbb{R}$.

Exercise 2

1. Given a map $f : \mathbb{R} \to \mathbb{R}$, the statement is '$f$ is continuous on $\mathbb{R}$'.

   Let $L = \{f_0\} \cup L_{ring}$, where f_0 is a unary function symbol. Let $M = \mathbb{R}$ with its natural L_{ring}-structure, where we interpret f_0 as the given map f.

   We define the abbreviation $a \geq b$ as $\exists x(a - b = x^2)$ and $a > b$ as $a \geq b \wedge a \neq b$.

   Let σ be the L-sentence

   $$\forall x_0 \forall \epsilon (\epsilon > 0 \to \exists \delta (\delta > 0 \wedge \forall x (\delta^2 > (x - x_0)^2 \to \epsilon^2 > (f_0(x) - f_0(x_0))^2))).$$

   Then, f is continuous on $\mathbb{R}$ if and only if $M \models \sigma$.

2. The set X has exactly 3 elements and the subset $Y \subseteq X$ has exactly 2 elements.

   Let $L = \{Y_0\}$, where Y_0 is a unary relation symbol. Let $M = X$, where we let $Y_0^M(x)$ if and only if $x \in Y$.

   We let σ be the L-sentence

   $$\exists a, b, c(a \neq b \wedge a \neq c \wedge b \neq c) \wedge \forall a, b, c, d(a = b \vee a = c \vee a = d \vee b = c \vee b = d \vee c = d)$$
   $$\wedge \exists a, b(Y_0(a) \wedge Y_0(b) \wedge a \neq b) \wedge \forall a, b, c(Y_0(a) \wedge Y_0(b) \wedge Y_0(c) \to a = b \vee a = c \vee b = c).$$

   Since the first part of the first line describes that X has at least 3 distinct elements and the second part of the first part describes that X has at most 3 distinct elements, and the first part of the second line describes that Y has at least 2 distinct elements and the second part of the second line describes that Y has at most 2 distinct elements, it follows that $M \models \sigma$ if and only if X contains exactly 3 distinct and Y contains exactly 2 elements.

3. The binary relation $\leq$ is a linear order on X.

   Let $L = \{\leq_0\}$ be the language containing only one binary relation symbol. Let $M = X$ and interpret $\leq_0$ as the relation given on X.

   Let σ be the L-sentence

   $$\forall x(x \leq_0 x) \wedge \forall x, y(x \leq_0 y \wedge y \leq_0 x \rightarrow x = y) \wedge \forall x, y, z(x \leq_0 y \wedge y \leq_0 z \rightarrow x \leq_0 z) \wedge \forall x, y(x \leq_0 y \vee y \leq_0 x).$$

   Then, $M \models \sigma$ if and only if $(X, \leq)$ is a linear order.

4. $\Gamma \subseteq X^2$ is the graph of a surjective function from X to X.

   Let $L = \{\Gamma_0\}$ the language where Γ_0 is a binary relation symbol. Let $M = X$ and we interpret Γ_0 as Γ.

   Let σ be the L-sentence

   $$\forall x \exists y \Gamma_0(x, y) \wedge \forall x \forall y, y'(\Gamma_0(x, y) \wedge \Gamma_0(x, y') \rightarrow y = y') \wedge \forall y \exists x \Gamma_0(x, y).$$

   The first and second part describe that Γ is the graph of a function: each $x \in X$ has one and only one image. The third part describes that this function is surjective.

5. Given a field K, the statement is 'every injective polynomial map from K to K is surjective'.

   Let $L = \{I_0, S_0\}$, where I and S are unary relation symbols. Let M be the set of polynomial maps from K to K and let I be the subset of M of all injective maps and S be the subset of M of all surjective maps. We let $I_0^M := I$ and $S_0^M := S$.

   Let σ be the L-sentence

   $$\forall x(I_0(x) \rightarrow S_0(x)).$$

   Then $M \models \sigma$ if and only if every injective polynomial map from K to K is surjective.

6. Given a field K, the statement is 'the polynomial $a_0 + a_1X + a_2X^2 + a_3X^3$ is irreducible over K'.

   Let $L = \{a_0, a_1, a_2, a_3\} \cup L_{ring}$, where $a_0, ..., a_3$ are constant symbols. Let $M = K$ naturally interpreted as an L_{ring}-structure and let a_i be interpreted as the i-th coefficient of the given polynomial.

   Let σ be the L-sentence

   $$(a_1 \neq 0 \wedge a_2 = 0 \wedge a_3 = 0) \vee ((a_2 \neq 0 \vee a_3 \neq 0) \wedge \forall x(a_0 + a_1x + a_2x^2 + a_3x^3 \neq 0)).$$

   Since the given polynomial has degree at most 3, it is irreducible if and only if it has degree 1 or it has degree 2 or 3 and has no roots in K.

7. Through every two distinct points there is exactly one straight line.

   Let $L = \{P, L, \in\}$, where P, L are unary relation symbols and $\in$ is a binary relation symbol. Let M be a set of points and lines, where we interpret $P(x)$ as 'x is a point' and $L(y)$ as 'y is a line' and $x \in y$ as 'x is a point on the line y', if x is a point and y is a line.

   Let σ be the L-sentence

   $$\forall x, x'(P(x) \wedge P(x') \wedge x \neq x' \rightarrow \exists y(L(y) \wedge x \in y \wedge x' \in y \wedge \forall y'(L(y') \wedge x \in y' \wedge x' \in y' \rightarrow y = y'))).$$

   Then $M \models \sigma$ if and only if there is one line through every two distinct points and this line is uniquely determined.

Exercise 3

Let M be an L-structure and N an L-substructure of M.

1. **Claim.** The inclusion map $i : N \hookrightarrow M$ is an L-embedding.

   Proof. Let c be a constant symbol. Then, $i(c^N) = c^N = c^M$, since N is a substructure. Let f be a function symbol and $\bar{a} \in N^{n_f}$. Then, $i(f^N(\bar{a})) = f^N(\bar{a}) = f^M(\bar{a}) = f^M(i(\bar{a}))$ since $f^N = f^M|_N$. Let r be a relation symbol and $\bar{a} \in N^{n_r}$. Since $r^N = r^M \cap N^{n_r}$, we have $r^N(\bar{a}) \Leftrightarrow r^M(\bar{a}) \Leftrightarrow r^M(i(\bar{a}))$, so i is an L-morphism and an L-embedding.

2. **Claim.** If $\varphi(x_1, ..., x_n)$ is a quantifier-free formula and $(a_1, ..., a_n)$ an n-tuple in N, then

$$N \models \varphi(a_1, ..., a_n) \Leftrightarrow M \models \varphi(a_1, ..., a_n).$$

   Proof. By induction on $c(\varphi)$.

   If φ is an atomic formula, it is of the form $r(t_1, ..., t_m)$ for a relation symbol r and terms $t_1, ..., t_n$. Then

$$N \models \varphi(a_1, ..., a_n) \Leftrightarrow (t_1^N(\bar{a}), ..., t_m^N(\bar{a})) \in r^N \Leftrightarrow (t_1^M(\bar{a}), ..., t_m^M(\bar{a})) \in r^M \Leftrightarrow M \models \varphi(a_1, ..., a_n),$$

   where we use that $t^N(\bar{a}) = t^M(\bar{a})$ for any term t (since the constants are interpreted the same way and the interpretation of a function in N is the restriction of the interpretation in M) and $r^N = r^M \cap N^{n_r}$.

   If φ is of the form $\neg\psi$ or $\psi_1 \wedge \psi_2$, the claim is clear by induction hypothesis. Since φ is quantifier-free, these are all cases.

3. **Claim.** If $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ is a quantifier-free formula and $\bar{a}$ an m-tuple in N, then

$$N \models \exists x_1, ..., x_n \varphi(x_1, ..., x_n, \bar{a}) \Rightarrow M \models \exists x1, ..., x_n \varphi(x_1, ..., x_n, \bar{a}).$$

   The converse is not true.

   Proof. Let $N \models \exists x_1, ..., x_n \varphi(x_1, ..., x_n, \bar{a})$. By definition, there are $b_1, ..., b_n \in N$ such that $N \models \varphi(\bar{b}, \bar{a})$. By 2., this implies $M \models \varphi(\bar{b}, \bar{a})$, so $M \models \exists x_1, ..., x_n \varphi(x_1, ..., x_n, \bar{a})$.

   For the converse, let $L = L_{ring}$ and $M = \mathbb{R}$ and $N = \mathbb{Z}$ with their usual interpretations. Then, $M \models \exists x(x^2 = 1 + 1)$, but $N \not\models \exists x(x^2 = 1 + 1)$.

4. **Claim.** If $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ is a quantifier-free formula and $\bar{a}$ an m-tuple in N, then

$$M \models \forall x_1, ..., x_n \varphi(x_1, ..., x_n, \bar{a}) \Rightarrow N \models \forall x_1, ..., x_n \varphi(x_1, ..., x_n, \bar{a}).$$

   Proof. Let $\bar{b} \in N^n \subseteq M^n$. Then $M \models \varphi(\bar{b}, \bar{a})$ by assumption, so by 2. we have $N \models \varphi(\bar{b}, \bar{a})$. Since $\bar{b}$ was arbitrary, we conclude $N \models \forall x_1, ..., x_n \varphi(x_1, ..., x_n, \bar{a})$.

   Also here, the converse does not hold; with notations as in 3., we have $N \models \forall x(x^2 \neq 1 + 1)$, but $M \not\models \forall x(x^2 \neq 1 + 1)$.

Exercise 4

Claim. Every L-formula is logically equivalent to a prenex one.

Proof. Let φ be an L-formula. Proof by induction on $c(\varphi)$.

If φ is an atomic formula, then φ is already prenex.

Let φ be of the form $\neg\psi$. By induction hypothesis, ψ is logically equivalent to a prenex formula ψ'. By induction on the number of quantifiers, we show that then $\neg\psi$ is logically equivalent to a prenex formula. If there are no quantifiers in ψ', we are done. Otherwise, let ψ' be of the form $Qx\psi''$, where Q is a quantifier. Then, $\neg\psi'$ is $\neg Qx\psi''$, which is logically equivalent to $Q^\vee x \neg\psi''$, where $Q^\vee$ is $\forall$ if Q is $\exists$ and vice versa. Since $\neg\psi''$ has one quantifier less than ψ', by induction hypothesis it follows that $\neg\psi$ is logically equivalent to a prenex formula.

Let φ be of the form $\psi_1 \wedge \psi_2$. By induction hypothesis, ψ_1 and ψ_2 are logically equivalent to prenex formulas ψ_1' respectively ψ_2'. Possibly after renaming variables we can assume that the bound variables in ψ_1' and ψ_2' are distinct from each other. Again by induction on the number of quantifiers in ψ_1' plus ψ_2', we can show that $\psi_1' \wedge \psi_2'$ is logically equivalent to a prenex formula: Let ψ_1' be $Qx\psi_1''$ and let ψ_2' be $Q'x'\psi_2''$. Then $\psi_1' \wedge \psi_2'$ is $Qx\psi_1'' \wedge Q'x'\psi_2''$, which is logically equivalent to $QxQ'x'(\psi_1'' \wedge \psi_2'')$ and by induction hypothesis, $\psi_1'' \wedge \psi_2''$ is logically equivalent to a prenex formula. Therefore, also ψ is logically equivalent to a prenex formula.

If φ is $\exists x\psi$, then ψ is logically equivalent to a prenex formula ψ', so φ is logically equivalent to $\exists x\psi'$, which is prenex.