Model theory
3. Formal proofs

Exercise 1 (tautologies)

1. Let A, B and C be sentential variables. Compute the truth functions $f_{A\vee B}, f_{A\to B}$ and $f_{A\leftrightarrow B}$ in terms of f_A and f_B and show that $A \vee \neg A$, $A \to (B \to A)$, $(\neg A \to A) \to A$, $(A \to B) \leftrightarrow (\neg B \to \neg A)$ and $((A \to B) \wedge (A \to (B \to C))) \to (A \to C)$ are tautologies.

2. Let $A(a_1, \ldots, a_n)$ be a sentential formula in sentential variables $a_1, \ldots, a_n$. Let L be a language, $\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x})$ L-formulas. Show that for all L-structure M and all $\bar{a}$ in M, one has

$$M \models A(\varphi_1, \ldots, \varphi_n)(\bar{a}) \iff f_A(\varphi_1^M(\bar{a}), \ldots, \varphi_n^M(\bar{a})) = 1.$$

3. Show that every L-tautology is universally true. Does the converse hold?

Exercise 2 (a few formal proofs) Let $\varphi_1, \ldots, \varphi_n, \varphi$ and ψ be formulas, Λ a set of formulas. Show the following implications.

1. (conjunction) If $\Lambda \vdash \{\varphi_1, \ldots, \varphi_n\}$, then $\Lambda \vdash \varphi_1 \wedge \cdots \wedge \varphi_n$.

2. (contrapositive) $\Lambda \vdash \varphi \to \psi$ if and only if $\Lambda \vdash \neg\psi \to \neg\varphi$.

3. (universal quantifier axiom) $\vdash \forall x_1 \varphi \to \varphi((t, x_2, \ldots, x_n))$ where $\varphi(x_1, \ldots, x_n)$ is a formula, t a term and the terms $(t, x_2, \ldots, x_n)$ are compatible with φ.

4. (universal quantifier rule) $\Lambda \vdash \varphi$ if and only if $\Lambda \vdash \forall x \varphi$.

5. (introduction of $\exists$) If x has no free occurence in ψ and $\Lambda \vdash \varphi \to \psi$, then $\Lambda \vdash \exists x \varphi \to \psi$.

6. $\vdash \forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$ for formulas $\varphi(x)$ and $\psi(x)$.

Exercise 3 (on the Deduction Lemma) Let Λ be a set of formulas, φ and ψ formulas. Does $\Lambda \vdash \varphi \to \psi$ imply $\Lambda \cup \{\varphi\} \vdash \psi$? Does $\Lambda \cup \{\varphi\} \vdash \psi$ imply $\Lambda \vdash \varphi \to \psi$?

Exercise 4 (counting formulas) A set A is *countable* if there is an injective map f from A to $\mathbf{N}$.

1. Show that $\mathbf{N} \times \mathbf{N}$ is countable.

2. Show that if A is a countable alphabet, then the set of finite words in this alphabet is countable.

3. Show that if L is a countable language and V a countable set of variables, then the set of L-formulas using variables in V is countable.