Model theory
3. Formal proofs (correction)

Exercise 1

Note that $f_A^2 = f_A$ holds for every sentential formula A, since $0^2 = 0$ and $1^2 = 1$.

Claim 1.1 For any sentential formulas A, B and C, one has

$$f_{A\vee B} = f_A + f_B - f_A \cdot f_B, \tag{1}$$

$$f_{A\to B} = 1 - f_A + f_A \cdot f_B, \tag{2}$$

$$f_{A\leftrightarrow B} = 1 - |f_A - f_B|. \tag{3}$$

Proof. $A \vee B$ is by definition the sentential formula $\neg(\neg A \wedge \neg B)$. Using the inductive definition of the truth function, one has

$$f_{A\vee B} = 1 - (1 - f_A)(1 - f_B) = f_A + f_B - f_A \cdot f_B.$$

$A \to B$ is by definition the sentential formula $B \vee \neg A$ so, by (1), one has

$$f_{A\to B} = f_B + (1 - f_A) - f_B \cdot (1 - f_A) = 1 - f_A + f_A \cdot f_B.$$

$A \leftrightarrow B$ is by definition $A \to B \wedge B \to A$, so

$$\begin{aligned} f_{A\leftrightarrow B} &= (1 - f_A + f_A \cdot f_B) \cdot (1 - f_B + f_A \cdot f_B) \\ &= 1 - f_A - f_B + 2 f_A \cdot f_B \\ &= 1 - f_A^2 - f_B^2 + 2 f_A \cdot f_B \\ &= 1 - (f_A - f_B)^2. \end{aligned}$$

As $(f_A - f_B)$ equals either 0, 1, or -1, one has $(f_A - f_B)^2 = |f_A - f_B|$. □

Claim 1.2 For any sentential formulas A, B and C, writting D, E, F, G and H for the sentential formulas $A \vee \neg A$, $A \to (B \to A)$, $(\neg A \to A) \to A$, $(A \to B) \leftrightarrow (\neg B \to \neg A)$ and $((A \to B) \wedge (A \to (B \to C))) \to (A \to C)$ respectively, one has

$$f_D = f_E = f_F = f_G = f_H = 1,$$

hence D, E, F, G and H are tautologies.

Proof. This is a direct application of Claim 1.1. □

Claim 1.3 Let $A(a_1, \ldots, a_n)$ be a sentential formula in sentential variables $a_1, \ldots, a_n$. Let L be a language, $\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x})$ L-formulas. For all L-structure M and all $\bar{a}$ in M, one has

$$M \models A(\varphi_1, \ldots, \varphi_n)(\bar{a}) \iff f_A(\varphi_1^M(\bar{a}), \ldots, \varphi_n^M(\bar{a})) = 1,$$

and every tautology is universally true, but $\forall x (x = x)$ is a universally true sentence that is not a tautology.

Proof. By induction on the complexity of A. **If $\mathbf{c(A) = 0}$,** A is a sentential variable a_i for some $1 \leqslant i \leqslant n$, so $A(\varphi_1, \dots, \varphi_n)$ is the formula φ_i, and $f_A(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a}))$ equals $\varphi_i^M(\bar{a})$. One has

$$M \models A(\varphi_1, \dots, \varphi_n)(\bar{a}) \iff M \models \varphi_i(\bar{a}) \iff \varphi_i^M(\bar{a}) = 1 \iff f_A(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a})) = 1.$$

If $\mathbf{A}$ is $\mathbf{B \wedge C}$, then $c(B) < c(A)$ and $c(C) < c(A)$ and one has $f_A = f_B \cdot f_C$ so

$$\begin{aligned} M \models A(\varphi_1, \dots, \varphi_n)(\bar{a}) &\iff M \models B(\varphi_1, \dots, \varphi_n)(\bar{a}) \text{ and } M \models C(\varphi_1, \dots, \varphi_n)(\bar{a}) \\ &\iff f_B(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a})) = 1 \text{ and } f_C(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a})) = 1 \\ &\iff (f_B \cdot f_C)(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a})) = 1 \\ &\iff f_A(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a})) = 1. \end{aligned}$$

If $\mathbf{A}$ is $\mathbf{\neg B}$, then one has $c(B) < c(A)$ and $f_A = 1 - f_B$ so

$$\begin{aligned} M \models A(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a})) &\iff M \models (\neg B)(\varphi_1, \dots, \varphi_n)(\bar{a}) \\ &\iff M \models \neg(B(\varphi_1, \dots, \varphi_n))(\bar{a}) \\ &\iff M \not\models B(\varphi_1, \dots, \varphi_n)(\bar{a}) \\ &\iff f_B(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a})) = 0 \\ &\iff (1 - f_B)(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a})) = 1 \\ &\iff f_A(\varphi_1^M(\bar{a}), \dots, \varphi_n^M(\bar{a})) = 1. \end{aligned}$$

This shows in particular that if $\varphi(x_1, \dots, x_n)$ is an L-tautology, then M satisfies $\varphi(\bar{a})$ for every $\bar{a}$ in M^n, so φ is universally true. Conversely the sentence $\forall x(x = x)$ is a logical axiom hence universally true. If it is of the form $A(\varphi_1, \dots, \varphi_n)$ for some sentential formula A, then A is either a sentential variable or the negation of a sentential variable and, in either case, A is not a tautology. ☐

Exercise 2

Let $\varphi_1, \dots, \varphi_n, \varphi$ and ψ be formulas, Λ a set of formulas.

Claim 2.1 If $\Lambda \vdash \{\varphi_1, \dots, \varphi_n\}$, then $\Lambda \vdash \varphi_1 \wedge \dots \wedge \varphi_n$.

Proof. Note that $\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)$ and $(\varphi_1 \wedge \varphi_2) \wedge \varphi_3$ are not the same formulas. However, since $A \wedge (B \wedge C) \to (A \wedge B) \wedge C$ and $(A \wedge B) \wedge C \to A \wedge (B \wedge C)$ are tautologies, proving $\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)$ is equivalent to proving $(\varphi_1 \wedge \varphi_2) \wedge \varphi_3$, so the claim $\vdash \varphi_1 \wedge \dots \wedge \varphi_n$ is not ambiguous. We prove the claim by induction on n. **If $\mathbf{n = 1}$**, the claim is obvious. **If $\mathbf{n = 2}$,** let $(\alpha_1, \dots, \alpha_k, \varphi_1)$ be a proof of φ_1 in Λ, and $(\beta_1, \dots, \beta_\ell, \varphi_2)$ a proof of φ_2 in Λ. Then

$$\Big(\alpha_1, \dots, \alpha_k, \varphi_1, \beta_1, \dots, \beta_\ell, \varphi_2, \varphi_1 \to (\varphi_2 \to (\varphi_1 \wedge \varphi_2)), \varphi_2 \to (\varphi_1 \wedge \varphi_2), \varphi_1 \wedge \varphi_2\Big)$$

is a proof of $\varphi_1 \wedge \varphi_2$ in Λ (the last three steps of the proof are obtained by applying the tautology $A \to (B \to (A \wedge B))$ and modus ponens twice). **For the induction step**, if Λ proves both $\varphi_1 \wedge \dots \wedge \varphi_{n-1}$ and φ_n, then it also proves $(\varphi_1 \wedge \dots \wedge \varphi_{n-1}) \wedge \varphi_n$ by the case $n = 2$. ☐

Claim 2.2 $\Lambda \vdash \varphi \to \psi$ if and only if $\Lambda \vdash \neg\psi \to \neg\varphi$.

Proof. If $(\alpha_1, \dots, \alpha_k, \varphi \to \psi)$ is proof of $\varphi \to \psi$ in Λ, then

$$\Big(\alpha_1, \dots, \alpha_k, \varphi \to \psi, (\varphi \to \psi) \to (\neg\psi \to \neg\varphi), \neg\psi \to \neg\varphi\Big)$$

is a proof of $\neg\psi \to \neg\varphi$ (where the last but one step is obtained using the tautology $(A \to B) \to (\neg B \to \neg A)$, and the last step by modus ponens). The reverse direction is similar using the tautolgy $(\neg B \to \neg A) \to (A \to B)$ instead. ☐

Claim 2.3 $\vdash \forall x_1\varphi \to \varphi((t, x_2, \ldots, x_n))$ where $\varphi(x_1, \ldots, x_n)$ is a formula, t is a term and the terms $(t, x_2, \ldots, x_n)$ are compatible with φ.

Proof. This is the contrapositive of the existential quantifier axiom. Let Λ be any set of formulas. As Λ proves $\neg\varphi((t, x_2, \ldots, x_n)) \to \exists x_1 \neg\varphi$ (this is an $\exists$-axiom), it also proves

$$\neg\exists x_1\neg\varphi \to \neg\neg\varphi((t, x_2, \ldots, x_n))$$

by Claim 2.2, that is

$$\forall x_1\varphi \to \neg\neg\varphi((t, x_2, \ldots, x_n)).$$

Using the tautologies $\neg\neg A \leftrightarrow A$ and $((A \to B) \wedge (B \leftrightarrow C)) \to (A \to C)$, and applying Claim 2.1, it follows that Λ proves

$$\forall x_1\varphi \to \varphi((t, x_2, \ldots, x_n)). \quad \square$$

Claim 2.4 $\Lambda \vdash \varphi$ if and only if $\Lambda \vdash \forall x\varphi$.

Proof. **Assume that $\mathbf{\Lambda}$ proves $\forall \mathbf{x}\varphi$** for a formula $\varphi(x_1, \ldots, x_n)$ and a variable x. Either x is among $x_1, \ldots, x_n$, say $x = x_1$, or it is not. In the last case, increasing n if necessary, one can view φ as a formula in variables $(x_1, \ldots, x_n)$ with $x = x_1$, and in both cases, the terms $(x_1, \ldots, x_n)$ are compatible with φ. Then by Claim 2.3 and modus ponens, Λ also proves $\varphi((x_1, \ldots, x_n))$, which is precisely φ. **Conversely, if $\mathbf{\Lambda}$ proves φ,** let $(\alpha_1, \ldots, \alpha_k, \varphi)$ be a proof. Then

$$\Big(\alpha_1, \ldots, \alpha_k, \varphi, \sigma, \varphi \to (\sigma \to \varphi), \sigma \to \varphi, \sigma \to \forall x\varphi, \forall x\varphi\Big)$$

is a proof of $\forall x\varphi$ (where σ is any logical axiom that is a **sentence** and where the 4 last steps of the proof are obtained using the tautology $A \to (B \to A)$, modus ponens, the generalisation rule, and modus ponens). $\square$

Claim 2.5 If x has no free occurence in ψ and $\Lambda \vdash \varphi \to \psi$, then $\Lambda \vdash \exists x\varphi \to \psi$.

Proof. This is the contrapositive of the generalisation rule. If Λ proves $\varphi \to \psi$, then it proves $\neg\psi \to \neg\varphi$ by Claim 2.2. As x does not have any free occurence in $\neg\psi$, by the generalisation rule, Λ proves $\neg\psi \to \forall x\neg\varphi$, that is $\neg\psi \to \neg\exists x\neg\neg\varphi$, hence $\exists x\neg\neg\varphi \to \psi$ by Claim 2.2 again. Using the axiom $\exists x\neg\neg\varphi \leftrightarrow \exists x\varphi$ and the tautolgy $((A \leftrightarrow B) \wedge B \to C) \to A \to C)$, one deduces that Λ proves $\exists x\varphi \to \psi$. $\square$

Claim 2.6 Let $\varphi(x)$ and $\psi(x)$ be formulas. Then $\vdash \forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$.

Proof. Note that the above formula is of the form $A \to (B \to C)$, that is $C \vee \neg\, B \vee \neg A$, that is $C \vee \neg(\neg\neg A \wedge \neg\neg B)$ hence $(\neg\neg A \wedge \neg\neg B) \to C$. It follows that $\Big((A \wedge B) \to C\Big) \to \Big(A \to (B \to C)\Big)$ is a tautology, so it suffices to show

$$\vdash (\forall x(\varphi \to \psi) \wedge \forall x\varphi) \to \forall x\psi. \tag{4}$$

Let σ be the sentence $(\forall x(\varphi \to \psi) \wedge \forall x\varphi))$, then $\{\sigma\}$ proves $\forall x(\varphi \to \psi)$ and $\forall x\varphi$ (using the tautology $A \wedge B \to A$), hence $\varphi \to \psi$ and φ by Claim 2.4, hence ψ by modus ponens, hence $\forall x\psi$ by Claim 2.4. We have shown

$$\{\sigma\} \vdash \forall x\psi.$$

By the Deduction Lemma, it follows that $\vdash \sigma \to \forall x\psi$. $\square$

Exercise 3

Claim 3.1 Let Λ be a set of formulas, φ and ψ formulas. $\Lambda \vdash \varphi \to \psi$ implies $\Lambda \cup \{\varphi\} \vdash \psi$, but $\Lambda \cup \{\varphi\} \vdash \psi$ does not imply $\Lambda \vdash \varphi \to \psi$.

Proof. **If Λ proves** $\varphi \to \psi$, then so does $\Lambda \cup \{\varphi\}$, so $\Lambda \cup \{\varphi\}$ proves ψ by modus ponens. **Conversely,** let Λ be the empty set, φ the formula $x = c$ and ψ the formula $\forall x(x = c)$ where c is a constant symbol. One has $\{x = c\} \vdash \forall x(x = c)$ by Claim 2.2. If one had $\vdash x = c \to \forall x(x = c)$, then $x = c \to \forall x(x = c)$ would be universally true (by the Theorem saying that a syntactic consequence is a semantic one). But the latter formula does not hold in the structure $\{0, 1\}$ having two distinct elements 0 and 1 where c is interpreted by 0. □

Exercise 4

Claim 4.1 $\mathbf{N} \times \mathbf{N}$ is countable. It follows that

0. If there is an injective map from A to B and B is countable, then A is countable.
1. If $A_1, \dots, A_n$ are countable, then $A_1 \times \cdots \times A_n$ is countable.
2. If $\{A_n : n \geqslant 1\}$ is a countable set of countable (not necess. disjoint) sets, $\bigcup_{n \geqslant 1} A_n$ is countable.

Proof. The map f from $\mathbf{N} \times \mathbf{N}$ to $\mathbf{N}$ mapping (n, m) to $2^n 3^m$ is injective by unicity of the prime factorisation.

0. If g is an injective map from A to B and h an injective map from B to $\mathbf{N}$, then $h \circ g$ is an injective map from A to $\mathbf{N}$.

1. It is enough to show it for $n = 2$, as the result for all n follows by a straightforeward induction on n. If $f : A_1 \to \mathbf{N}$, $g : A_2 \to \mathbf{N}$ and $i : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ are injective maps, then $(x, y) \to i(f(x), g(y))$ is an injective map from $A_1 \times A_2$ to $\mathbf{N}$.

2. Let $f_n : A_n \to \mathbf{N}$ be an injective map for every $n \geqslant 1$. For every $x \in \bigcup_{n \geqslant 1} A_n$, let $m(x) \geqslant 1$ be the least natural number such that $x \in A_m$ (there can be several ones since the sets A_n are not assumed to be disjoint, and there is a least one as $\mathbf{N}$ is well-ordered). Then the map from $\bigcup_{n \geqslant 1} A_n$ to $\mathbf{N} \times \mathbf{N}$ mapping x to $(m(x), f_m(x))$ is an injective map, so $\bigcup_{n \geqslant 1} A_n$ is countable by 0. □

Claim 4.2 Let A be a countable alphabet. The set of finite words in this alphabet (*i.e.* of finite ordered subsets of A, or of finite tuples of A) is countable.

Proof. For every natural number $n \geqslant 1$, the set of words of length n is precisely A^n, and $\bigcup_{n \geqslant 1} A^n$ is countable by the above claim. □

Claim 4.3 If L is a countable language and V a countable set of variables, then the set of L-formulas using variables in V is countable.

Proof. An L-formula is a word in the alphabet $A = L \cup V \cup \{=, \wedge, \neg, \exists\}$, which is a countable set by Claim 4.1.2, so the set of L-formulas is countable by Claim 4.2. One could alternatively have taken an injective map $f : A \to \mathbf{N}$, an infinite set of pairwise distinct primes $\{p_n : n \geqslant 1\}$ and consider the function mapping a formula $(a_1, \dots, a_n)$ to the product $p_1^{f(a_1)} \cdots p_n^{f(a_n)}$. □