Model theory
4. Cartesian products and reduced products

Exercise 1 (expressing mathematical statements by sentences) For each statement S below, find a suitable language L, an L-structure (M, L^M) and an L-sentence σ such that S is equivalent to $M \models \sigma$.

1. Given natural numbers m, n, p, the statement is 'p is a prime number and m and n are coprime'.
2. Given a field K with a linear ordering on K, the statement is 'K is an ordered field'.
3. Given an ordered field K, the statement is 'every positive element is a square'.
4. Given a matrix $A = (a_{11}, a_{12}, a_{21}, a_{12})$ in $\mathrm{M}_2(\mathbf{R})$, the statement is 'A is invertible'.
5. Given a group G, the statement is 'the centre of G is non-trivial'.

Exercise 2 (satisfaction in a Cartesian product) Let $(M_i)_{i\in I}$ be a family of L-structures, $\varphi(x_1, \ldots, x_n)$ an atomic formula and $a^1, \ldots, a^n$ elements of $\prod_{i\in I} M_i$.

1. Show that $\prod_{i\in I} M_i$ satisfies $\varphi(a^1, \ldots, a^n)$ if and only if M_i satisfies $\varphi(a^1_i, \ldots, a^n_i)$ for all $i \in I$.
2. Does that hold for any formula?
3. Show that if $J \subset I$, the restriction map $\prod_{i\in I} M_i \longrightarrow \prod_{j\in J} M_j$ is a morphism.

Exercise 3 (building ultrafilters with prescribed elements) Let I and $J \subset I$ be infinite sets.

1. Show that there is a non-principal ultrafilter on I that contains $\{J\}$.
2. Is there a non-principal ultrafilter on $\mathbf{N}$ containing $\{n\mathbf{N} : n \geqslant 1\}$?
3. Under which conditions on a set $\mathcal{G}$ of subsets of I is there a non-principal ultrafilter extending $\mathcal{G}$?

Exercise 4 (product reduced by a principal ultrafilter) Let I be a set, $J \subset I$ a subset and $\mathcal{F}$ the principal filter on I generated by the singleton $\{J\}$. Let (M_i) a family of L-structures. Show that the reduced product $\prod_{\mathcal{F}} M_i$ is isomorphic to the Cartesian product $\prod_{j\in J} M_j$.

Exercise 5 (reduced product of rings) Consider $\mathbf{R}$ with its L_{ring}-structure. The L_{ring}-structure $\mathbf{R}^{\mathbf{N}}$ is the natural ring structure on the Cartesian power of $\mathbf{R}$. Let $\mathcal{F}$ be a filter on $\mathbf{N}$.

1. Show that there is an ideal $I_{\mathcal{F}}$ of $\mathbf{R}^{\mathbf{N}}$ such that $\mathbf{R}^{\mathcal{F}}$ is precisely the quotient ring $\mathbf{R}^{\mathbf{N}}/I_{\mathcal{F}}$.
2. If $\mathcal{F}$ is not an ultrafilter, show that $\mathbf{R}^{\mathcal{F}}$ is not a field. What can you say about the L_{ring}-theory of $\mathbf{R}^{\mathcal{F}}$, using either 1. or the particular cases where Łos' Theorem holds for a reduced product? Can you explain why Łos Theorem fails to conclude that $\mathbf{R}^{\mathcal{F}}$ is a field?
3. If $\mathcal{U}$ is an ultrafilter on $\mathbf{N}$, show that the ideal $I_{\mathcal{U}}$ is maximal. What can you say about the L_{ring}-theory of $\mathbf{R}^{\mathcal{U}}$?
4. Conversely, show that for every ideal I of $\mathbf{R}^{\mathbf{N}}$, there is a filter $\mathcal{F}$ on $\mathbf{N}$ such that $\mathbf{R}^{\mathbf{N}}/I$ equals $\mathbf{R}^{\mathcal{F}}$.