Model theory
4. Cartesian and reduced products (correction)

Exercise 1

1. One can take the language $\{+, \times, 1, a, b, c\}$, the structure $\mathbf{N}$ (with $+$, $\times$ and 1 having their natural interpretion in $\mathbf{N}$ and the constant symbols a, b and c interpreted by m, n and p) and consider the sentence
$$\Big(\forall x \forall y (xy = p \to (x = 1 \lor y = 1))\Big) \land \exists u \exists v (um + vn = 1).$$
2. One can take the language of rings $\{+, \times, -, 0, 1, \leqslant\}$ augmented with a binary relation symbol, the natural L_{ring}-structure on K and $\leqslant^K$ the given linear ordering on K and the sentence
$$\forall x \forall y \forall z (x \leqslant y \to x + y \leqslant x + z) \land \big((x \geqslant 0 \land y \geqslant 0) \to xy \geqslant 0\big).$$
3. One can take the language of rings $\{+, \times, -, 0, 1, \leqslant\}$ augmented with a binary relation symbol, the natural L_{ring}-structure on K and $\leqslant^K$ the given linear ordering on K and the sentence
$$\forall x (x \geqslant 0 \to \exists y (x = y^2)).$$
4. One can take the language of rings $\{+, \times, -, 0, 1, \leqslant\}$ augmented with 4 constant symbols a, b, c, d, the natural L_{ring}-structure on the field $\mathbf{R}$ of real numbers (with $a_{11}, a_{12}, a_{21}, a_{22}$ interpreting a, b, c, d) and the sentence
$$\exists x \exists y \exists s \exists t (ax + bz = 1 \land cy + dt = 1 \land ay + bt = 0 \land cx + dz = 0),$$
or even the quantifier-free sentence
$$ad - bc = 0.$$
5. One can take the language of groups, the natural L_{gp}-structure on G and the sentence
$$\exists x (\forall y (yx = xy) \land x \neq 1).$$

Exercise 2

Let $(M_i)_{i \in I}$ be L-structures $\varphi(x_1, \ldots, x_n)$ an atomic formula and $a^1, \ldots, a^n$ elements of $\prod_i M_i$.

Claim 2.1
$$\prod_i M_i \models \varphi(a^1, \ldots, a^n) \iff M_i \models \varphi(a^1_i, \ldots, a^n_i) \text{ for all } i \in I$$

Proof. One can show first that, if $t(\bar{x})$ is an L-term, writing M the Cartesian product $\prod_i M_i$, then for all $a^1, \ldots, a^n$ in M, one has $t^M(a^1, \ldots, a^n) = (t^{M_i}(a^1_i, \ldots, a^n_i))_{i \in I}$ (by induction on the complexity $c(t)$ applying the definition of the interpretation of a constant symbol c and function symbol in $\prod_i M_i$). The Claim follows by applying the definition of the interpretation of a relation symbol r in $\prod_i M_i$. $\square$

Claim 2.2 Claim 2.1 does not hold for any formula.

Proof. Consider for instance the language L with equality only, a set A with two distinct elements, the Cartesian product $A \times A$ and σ the sentence $\exists x \exists y \forall z((x \neq y) \wedge (z = x \vee z = y))$ stating that there are exactly two elements. $\square$

Claim 2.3 For any $J \subset I$, the restriction map $\alpha : \prod_{i \in I} M_i \longrightarrow \prod_{j \in J} M_j$ is an L-morphism.

Proof. Let us write M_I for $\prod_{i \in I} M_i$ and M_J for $\prod_{i \in J} M_i$. Let c, r and f be a constant symol, an n-ary relation symbol and an n-ary function symbol respectivel. One has

$$\alpha(c^{M_I}) = \alpha((c^{M_i})_{i \in I}) = (c^{M_i})_{i \in J} = c^{M_J},$$

$$\alpha(f^{M_I}(a^1, \ldots, a^n)) = \alpha((f^{M_i}(a^1_i, \ldots, a^n_i))_{i \in I}) = (f^{M_i}(a^1_i, \ldots, a^n_i))_{i \in J} = f^{M_J}(\alpha(a^1), \ldots, \alpha(a^n)), \quad \text{and}$$

$$\begin{aligned} (a^1, \ldots, a^n) \in r^{M_I} &\iff (\forall i \in I)\ (a^1_i, \ldots, a^n_i) \in r^{M_i} \\ &\implies (\forall i \in J)\ (a^1_i, \ldots, a^n_i) \in r^{M_i} \\ &\iff ((a^1_i)_{i \in J}, \ldots, (a^n_i)_{i \in J}) \in r^{M_J} \\ &\iff (\sigma(a^1), \ldots, \sigma(a^n)) \in r^{M_J}. \end{aligned}$$ $\square$

Note that Claim 2.1 actually holds for every *positive* formula (i.e. that does not use the negation symbol), using the Axiom of Choice.

Exercise 3

Let I and $J \subset I$ be infinite sets.

Claim 3.1 There is a non-principal ultrafilter on I that contains $\{J\}$.

Proof. Let $\mathcal{F}$ be the Fréchet filter on I. Any finitely many elements of $\mathcal{F} \cup \{J\}$ have a non-empty intersection, so $\mathcal{F} \cup \{J\}$ generates a filter on I, which can be extended to an ultrafilter $\mathcal{U}$ on I. As $\mathcal{U}$ contains the Fréchet filter, $\mathcal{U}$ is non-principal. $\square$

Claim 3.2 There is a non-principal ultrafilter on $\mathbf{N}$ containing $\{n\mathbf{N} : n \geqslant 1\}$.

Proof. Let $\mathcal{F}$ be the Fréchet filter on $\mathbf{N}$. Any finitely many elements of $\mathcal{F} \cup \{n\mathbf{N} : n \geqslant 1\}$ have a non-empty intersection, so $\mathcal{F} \cup \{n\mathbf{N} : n \geqslant 1\}$ generates a filter on $\mathbf{N}$, which can be extended to an ultrafilter $\mathcal{U}$ on $\mathbf{N}$. As $\mathcal{U}$ contains the Fréchet filter, $\mathcal{U}$ is non-principal. $\square$

Claim 3.3 Let $\mathcal{G}$ be a set of subsets of I. There is a non-principal ultrafilter extending $\mathcal{G}$ if and only if $G_1 \cap \cdots \cap G_n$ is infinite for every $G_1, \ldots, G_n$ in $\mathcal{G}$.

Proof. If $G_1 \cap \cdots \cap G_n$ is infinite for every $G_1, \ldots, G_n$ in $\mathcal{G}$, then the above argument holds: $\mathcal{G} \cup \mathcal{F}$ can be extended to an ultrafilter. Conversely, any non-principal ultrafilter $\mathcal{U}$ containing $\mathcal{G}$ must contain the Fréchet filter $\mathcal{F}$, and if $G_1 \cap \cdots \cap G_n$ was finite for some $G_1, \ldots, G_n$ in $\mathcal{G}$, one would have $I \setminus (G_1 \cap \cdots \cap G_n) \in \mathcal{F}$ so $(I \setminus (G_1 \cap \cdots \cap G_n)) \cap G_1 \cap \cdots \cap G_n = \emptyset \in \mathcal{U}$, a contradiction. $\square$

Exercise 4

Let I be a set, $J \subset I$ a subset and $\mathcal{F}$ the principal filter on I generated by the singleton $\{J\}$. Let $(M_i)_{i\in I}$ be a family of L-structures.

Claim 4.1 The reduced product $\prod_{\mathcal{F}} M_i$ is isomorphic to the Cartesian product $\prod_{j\in J} M_j$.

Proof. Let $\alpha : \prod_{\mathcal{F}} M_i \longrightarrow \prod_{j\in J} M_j$ be the map sending $\big((a_i)_{i\in I}\big)_{\mathcal{F}}$ to $(a_j)_{j\in J}$. We claim that α is well-defined and an L-isomorphism. If $\big((a_i)_{i\in I}\big)_{\mathcal{F}} = \big((b_i)_{i\in I}\big)_{\mathcal{F}}$, then the set $\{i \in I : a_i = b_i\}$ contains J, so $(a_i)_{i\in J} = (b_i)_{i\in J}$ and α is **well-defined**. If $(a_i)_{i\in J}$ is an element of $\prod_{j\in J} M_j$, let $(a_i)_{i\in I}$ in $\prod_{i\in I} M_i$ where a_i is abitrarily chosen in M_i for $i \in I \setminus J$ (using the axiom of choice). Then α maps $\big((a_i)_{i\in I}\big)_{\mathcal{F}}$ to $(a_j)_{j\in J}$, so α is **surjective**. Let us show that α is an **embedding**. Let c, f and r be a constant symbol, an n-ary function symbol and and n-ary relation symbol. We write $M_{\mathcal{F}}$ for $\prod_{\mathcal{F}} M_i$ and M_J for $\prod_{j\in J} M_j$. For every n-tuple $(a^1, \ldots, a^n)$ in $\prod_{i\in I} M_i$, one has

$$\alpha(c^{M_{\mathcal{F}}}) = \big((c^{M_i})_{i\in I}\big)_{\mathcal{F}} = (c^{M_i})_{i\in J} = c^{M_J},$$

$$\alpha(f^{M_{\mathcal{F}}}(a^1_{\mathcal{F}}, \ldots, a^n_{\mathcal{F}})) = \alpha\Big(\big((f^{M_i}(\bar{a}_i))_{i\in I}\big)_{\mathcal{F}}\Big) = \big(f^{M_j}(a^1_j, \ldots, a^n_j)\big)_{j\in J} = f^{M_J}(\alpha(a^1_{\mathcal{F}}), \ldots, \alpha(a^n_{\mathcal{F}})), \text{ and}$$

$$\begin{aligned}
(a^1_{\mathcal{F}}, \ldots, a^n_{\mathcal{F}}) \in r^{M_{\mathcal{F}}} &\iff \Big\{i \in I : (a^1_i \ldots, a^n_i) \in r^{M_i}\Big\} \in \mathcal{F} \\
&\iff J \subset \Big\{i \in I : (a^1_i, \ldots, a^n_i) \in r^{M_i}\Big\} \\
&\iff (\forall j \in J)\ (a^1_j, \ldots, a^n_j) \in r^{M_j} \\
&\iff (\alpha(a^1_{\mathcal{F}}), \ldots, \alpha(a^n_{\mathcal{F}})) \in r^{M_J}.
\end{aligned}$$

□