Model theory
5. Ultraproduct and the Compactness Theorem (correction)

Exercise 1

Claim 1.1 There is a group H that has the same L_{gp}-theory as G and has infinitely many elements of infinite order (using an ultraproduct construction).

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbf{N}$. The ultrapower $G^{\mathcal{U}}$, written H, is an L_{gp}-structure that satisfies the same L_{gp}-sentences as G according to Łos Theorem. In particular it is a group. For every natural number n, let g_n be an element of G of order at least n. Then if h denotes the element $(g_0, g_1, \dots, g_n, \dots)_{\mathcal{U}}$, one has for every natural number n,

$$h^n = (g_0^n, g_1^n, \dots, g_n^n, \dots)_{\mathcal{U}} \neq (1^G, 1^G, \dots, 1^G, \dots)_{\mathcal{U}}$$

for otherwise $\mathcal{U}$ would contain the finite set $\left\{k \in \mathbf{N} : g_k^n = 1^G\right\}$ for some n, contradicting the fact that $\mathcal{U}$ is non-principal. The powers of h form an infinite set of elements having infinite order. □

Claim 1.2 There is a group H that has the same L_{gp}-theory as G and has infinitely many elements of infinite order (using the Compactness Theorem).

Proof. Let c be a new constant symbol and consider the $L_{gp} \cup \{c\}$-theory

$$\Sigma = \Sigma(G) \cup \left\{c^n \neq 1 : n \in \mathbf{N}\right\}.$$

By assumption, G is a model of every finite subset $\Sigma_0 \subset \Sigma$ (one interprets c by an element of G of order greater than any natural number n occuring in Σ_0). By the Compactness Theorem, Σ has a model $(H, L_{gp}^H \cup c^H)$. As H satisfies $\Sigma(G)$, it is a group, and c^H has infinite order. □

Claim 1.3 There is no L_{gp}-formula $\varphi(x)$ that satisfies for all model M of $\Sigma(G)$ and element a of M the equivalence

$$M \models \varphi(a) \iff \text{the order of } a \text{ is finite.}$$

First proof. If $\varphi(x)$ was such a formula, by Łos Theorem, with the same notations as in Claim 1.1, one would have $G^{\mathcal{U}} \models \varphi(h)$ so h would have finite order. □

Second proof. If $\varphi(x)$ was such a formula, the $L \cup \{c\}$-theory

$$\Sigma(G) \cup \left\{\varphi(c)\right\} \cup \left\{c^n \neq 1 : n \in \mathbf{N}\right\}$$

would be finitely satisfiable (by G, interpreting c by an element of G of order greater than any natural number n occuring in the finite subset of Σ considered) hence satisfiable by some structure $(H, L_{gp}^H \cup c^H)$, so c^H would have both finite order (as $H \models \varphi(c)$) and infinite order, a contradiction. □

Exercise 2

Claim 2.1 The ultraproduct $\prod_{\mathcal{U}} F_p$ is a field having characteristic either 0 (when $\mathcal{U}$ is non-principal) or q (when $\mathcal{U}$ is principal, generated by $\{q\}$).

Proof. $\prod_{\mathcal{U}} F_p$ is an ultraproduct of L_{field}-structures, hence an L_{field}-structure. By Łos' Theorem, $\prod_{\mathcal{U}} F_p$ satisfies all the sentences that are true in every field F_p, and in particular, it is a field. If the ultrafilter $\mathcal{U}$ is principal, it is generated by a singleton $\{q\}$ for some prime number q, and we saw in the previous Exercise sheet that $\prod_{\mathcal{U}} F_p$ and F_q are isomorphic L_{field}-structures, so $\prod_{\mathcal{U}} F_p$ has characteristic q. If $\mathcal{U}$ is non-principal, it contains any cofinite set, and in particular the set

$$\left\{p \in \mathcal{P} : 0_{F_p} \neq 1^{F_p} + \cdots + 1^{F_p} \ (k \ times)\right\}$$

for any natural number k. By Łos' Theorem, if $1_{\mathcal{U}}$ denotes the element $(1^{F_2}, 1^{F_3}, 1^{F_5}, \ldots, 1^{F_p}, \ldots)_{\mathcal{U}}$ and $0_{\mathcal{U}}$ the element $(0^{F_2}, 0^{F_3}, 0^{F_5}, \ldots, 0^{F_p}, \ldots)_{\mathcal{U}}$, it follows that

$$0_{\mathcal{U}} \neq 1_{\mathcal{U}} + \cdots + 1_{\mathcal{U}} \ (k \ times),$$

for every k, so that $\prod_{\mathcal{U}} F_p$ has characteristic 0. □

Claim 2.2 Assume σ is an L_{field}-sentence and that for infinitely many prime numbers p, there is a field of characteristic p satisfying σ. There is a field of characteristic 0 that satisfies σ.

First proof. Let $\mathcal{P}$ be the infinite set of prime numbers satisfying the assumption and for every p in $\mathcal{P}$, let F_p be a field of characteristic p that satisfies σ. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathcal{P}$. By the previous Claim, $\prod_{\mathcal{U}} F_p$ is a field of characteristic 0. By Łos' Theorem, $\prod_{\mathcal{U}} F_p$ satifies σ. □

Second proof. Let Σ be the set of fields axioms. The L_{field}-theory

$$\Sigma \cup \left\{\sigma\right\} \cup \left\{0 \neq 1 + \cdots + 1 \ (k \ times) : k \in \mathbf{N}\right\}$$

is finitely satisfiable by assumption, hence satisfiable by the Compactness Theorem, by an L_{field}-structure K, which is a field (as it satisfies Σ) of characteristic 0. □

Claim 2.3 Assume that σ is an L_{field}-sentence that holds for every field of characteristic 0. There is a natural number n such that σ holds in every field of characteristic $p > n$.

Proof. By contrapositive. If for all n, there exists a field F_p of characteristic $p > n$ such that F_p does not satisfy σ, then $\neg\sigma$ holds in a field of characteristic p for infinitely many prime numbers p, so $\neg\sigma$ holds in a field of characteristic 0 by the previous claim. □

Exercise 3

Claim 3.1 Let $\mathbf{R}$ be equipped with its natural $L_{ring} \cup \{<\}$-structure. For any language L, $\mathbf{R}$ can be expanded as an $L \cup L_{ring} \cup \{<\}$-structure. If $\mathcal{U}$ is a non-principal ultrafilter on $\mathbf{N}$, then $\mathbf{R}^{\mathcal{U}}$ is an $L \cup L_{ring} \cup \{<\}$-structure having the same $L \cup L_{ring} \cup \{<\}$-theory as $\mathbf{R}$, and $\mathbf{R}^{\mathcal{U}}$ is a non-Archimedian ordered field.

Proof. Define $c^{\mathbf{R}}$ to be 1 for any constant symbol c, $f^{\mathbf{R}}$ to be the constant function 1 for any n-ary function symbol f, and $r^{\mathbf{R}}$ to be $\mathbf{R}^n$ for any n-ary relation symbol r. The ultrapower $\mathbf{R}^{\mathcal{U}}$ is an $L \cup L_{ring} \cup \{<\}$-structure, and by Łos' Theorem, an ordered field that satisfies the same theory as $\mathbf{R}$. The element $\left(1, 2, 3, \ldots, n, \ldots\right)_{\mathcal{U}}$ is greater than any $(n, n, \ldots, n, \ldots)_{\mathcal{U}}$ by Łos' Theorem, so $\mathbf{R}^{\mathcal{U}}$ is non-Archimedian. □

Claim 3.2 There exists a language L and an L-sentence σ such that for all ordered field K, there is an L-structure L^K on K, such that $K \models \sigma$ if and only if K is Archimedian.

Proof. Note that an ordered field must have characteristic 0 hence contains a copy of the natural numbers. Let P be a unary predicate (i.e. a unary relation symbol) that we interpret as the subset $\{n \cdot 1_K : n \in \mathbf{N}\}$ of K and σ the formula $\forall x \exists y (P(y) \wedge x \leqslant y)$. $\square$

Claim 3.3 For all language L (expanding the language of ordered fields), there is no L-sentence σ such that there is an L-structure $L^{\mathbf{R}}$ on $\mathbf{R}$ (expanding the usual ordered field structure on $\mathbf{R}$), such that for all models K of $\Sigma(\mathbf{R})$, one has $K \models \sigma$ if and only if K is Archimedian.

Proof. Let L be a language expanding the language of ordered fields. If there exists a sentence σ and an L-structure $L^{\mathbf{R}}$ on $\mathbf{R}$ (expanding the usual order field structure on $\mathbf{R}$) such that $K \models \sigma$ iff K is Archimedian for all model K of $\Sigma(\mathbf{R})$, then $(\mathbf{R}, L^{\mathbf{R}}) \models \sigma$. For any non-principal ultrafilter $\mathcal{U}$ on $\mathbf{N}$, the L-structure $\mathbf{R}^{\mathcal{U}}$ also satisfies σ by Łos' Theorem, but $\mathbf{R}^{\mathcal{U}}$ is not Archimedian by Claim 3.1. $\square$