# Model theory
# 6. Ordinals and cardinals

**Exercise 1** (ordinal arithmetic) 1. Show that if $\alpha$ is an ordinal, then $\alpha \cup \{\alpha\}$ is also an ordinal.

2. Show that $\omega^2$ and $\omega^\omega$ are countable ordinals.

3. Show that if $\alpha$ is an ordinal number obtained by finitely many applications of ordinal operations (addition, multiplication and exponentiation) to $\omega$ or natural numbers, then $\alpha$ is countable.

**Exercise 2** (Cantor normal form) Let $\alpha, \beta, \gamma$ be ordinals. Show the following.

1. If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$ (does $\beta + \alpha < \gamma + \alpha$ also hold?).

2. If $\alpha < \beta$, then there exists a unique ordinal $\delta$ such that $\alpha + \delta = \beta$.

3. (Euclidian division) If $\alpha > 0$ and $\gamma$ is arbitrary, then there exist a unique ordinal $\beta$ and a unique ordinal $\rho < \alpha$ such that $\gamma = \alpha \cdot \beta + \rho$.

4. (writing in base $\omega$) Every ordinal $\alpha > 0$ can be represented uniquely in the form

$$\alpha = \omega^{\beta_1}.n_1 + \cdots + \omega^{\beta_k} \cdot n_k,$$

where $n \geqslant 1$, $\alpha \geqslant \beta_1 > \cdots > \beta_n$ are ordinals and $n_1, \ldots, n_k$ are non-zero natural numbers.

**Exercise 3** (cardinal arithmetic) 1. Show that the cardinal addition $\kappa + \lambda$, multiplication $\kappa \cdot \lambda$ and exponentiation $\kappa^\lambda$ are well-defined, that $+$ and $\cdot$ are associative and that $\cdot$ is distributive over $+$.

2. If $X$ is any set, show that its power set $\mathcal{P}(X)$ has cardinality $2^{|X|}$.

3. Show that $\kappa^{\lambda+\mu} = \kappa^\lambda \kappa^\mu$ holds for any cardinal numbers $\lambda, \kappa$ and $\mu$.

4. Show that $\lambda + \lambda = \lambda$ for every infinite cardinal $\lambda$.

5. Show that $\lambda \cdot \lambda = \lambda$ for every infinite cardinal $\lambda$.

6. What is the cardinality of the set of finite subsets of $\lambda$?

**Exercise 4** (computing cardinals) 1. Show that the cardinality of irrational real numbers is $2^{\aleph_0}$.

2. Let $K/\mathbf{Q}$ be a field extension of $\mathbf{Q}$. Show that the set of elements of $K$ that are algebraic over $\mathbf{Q}$ is countable. Let $K/F$ be any field extension. What can you say about the cardinality of the set of elements of $K$ that are algebraic over $F$?

3. Show that the cardinality of the set of real transcendental numbers is $2^{\aleph_0}$.

4. Let $K$ be any field and $V$ an infinite $K$-vector space with basis $B$. Show that $|K| + |B| = |V|$.