Model theory
7. Elementary substructures and extensions

Exercise 1 (two elementary equivalent structures embed elementarily in a common structure) Let N and M be two L-structures, $\sigma : N \longrightarrow M$ any map from N to M, and M_σ the $L \cup N$-structure $(M, L^M \cup N^M)$ obtained by interpreting every constant symbol n in N by $\sigma(n)$. In the particular case when the map σ is the identity map id_M from M to M, the $L \cup M$-theory $\Sigma(M_{id_M})$ is called the *elementary diagram of* M, written $\Delta_e(M)$.

1. Show that the map $\sigma : N \longrightarrow M$ is an elementary embedding if and only if $M_\sigma \models \Delta_e(N)$.

2. Assume that M and N are elementarily equivalent. Show that $\Delta_e(N) \cup \Delta_e(M)$ is a satisfiable $L \cup N \cup M$-theory (we assume that the sets N and M are disjoint). Deduce that there exists an L-structure K in which both M and N embed elementarily.

Exercise 2 (an application of Löwenheim-Skolem Theorem to simple groups) A group G is said to be *simple* if G and $\{1\}$ are its only normal subgroups. Let G be an infinite simple group. The aim of the exercise is to show that for every infinite cardinal $\kappa \leqslant |G|$, the group G has a simple subgroup of cardinality κ.

1. Let Σ be any L-theory having an infinite model. For every infinite cardinal $\kappa \geqslant |L|$, show that there exists a model of Σ of cardinality κ.

2. Show that every elementary L_{gp}-substructure of G is a simple group, and conclude.

Exercise 3 (two model companions of Σ have the same models)

1. Let I be a linear ordering and $(M_i)_{i \in I}$ a family of L-structures such that $M_i \prec M_j$ whenever $i \leqslant j$. Show that $\bigcup_{i \in I} M_i$ is an elementary extension of M_i for every i in I.

2. Let Σ be an L-theory, and Σ_1 and Σ_2 two model companions of Σ. Show that Σ_1 and Σ_2 have the same models.

Exercise 4 (an alternative proof that ACF is model complete) Let Σ be the L_{ring}-theory of algebraically closed fields and M, N two models of Σ.

1. Recall briefly what are the atomic L_{ring}-formulas and their interpretations in a model of Σ.

2. Let $\bar{a}$ in M and $\bar{b}$ in N be two n-tuples such that for any atomic L_{ring}-formula $\varphi(\bar{x})$, one has
$$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{b}). \tag{1}$$
Show that (1) holds for any L_{ring}-formula $\varphi(\bar{x})$.

3. Conclude that the theory of all algebraically closed fields is model complete.