Model theory
7. Elementary substructures and extensions (correction)

Exercise 1

Claim 1.1 Let (M, L^M) be an L-structure, $\varphi(\bar{x})$ a formula and $\bar{a}$ a tuple from M. Let us consider the language $L \cup \bar{b}$, where we have added to L a new constant symbol b_i for each coordinate of $\bar{a}$. Consider the $L \cup \bar{b}$-structure $M' = (M, L^M, \bar{b})$ (i.e. every element of L has the same interpretation as in M, and b_i is interpreted by a_i). Then

$$M \models \varphi(\bar{a}) \iff M' \models \varphi((\bar{b})). \tag{1}$$

Note that $\varphi(\bar{x})$ is an L-formula, and $\varphi((\bar{a}))$ is an $L \cup \bar{a}$-sentence.

Proof. One can show easily by induction on the complexity of an L-term $t(\bar{x})$ that $t^M(\bar{a}) = t((\bar{b}))^{M'}$ for every L-term $t(\bar{x})$, using the fact that $f^M = f^{M'}$ for every function symbol f of L. We show (1) by induction on the complexity of φ. If $\varphi(\bar{x})$ is the atomic formula $r(t_1(\bar{x}), \ldots, t_n(\bar{x}))$, then

$$\begin{aligned} M \models \varphi(\bar{a}) &\iff (t_1^M(\bar{a}), \ldots, t_n^M(\bar{a})) \in r^M \\ &\iff (t_1((\bar{b}))^{M'}, \ldots, t_n((\bar{b}))^{M'}) \in r^M \\ &\iff (t_1((\bar{b}))^{M'}, \ldots, t_n((\bar{b}))^{M'}) \in r^{M'} \\ &\iff M' \models \varphi((\bar{b})). \end{aligned}$$

If (1) is proved for ψ_1 and ψ_2, it clearly also holds for $\neg\psi_1$ and $\psi_1 \wedge \psi_2$. If φ is of the form $\exists y \psi(y, \bar{x})$ and if (1) holds for ψ, as M and M' have the same domain, the claim holds for $\exists y \psi(y, \bar{x})$. □

Claim 1.2 The map $\sigma : N \longrightarrow M$ is an elementary embedding if and only if $M_\sigma \models \Delta_e(N)$.

Proof. By definition, $\sigma : N \longrightarrow M$ is an elementary embedding if and only if for all $\varphi(\bar{x})$ and $\bar{n} \in N$,

$$N \models \varphi(\bar{n}) \iff M \models \varphi(\sigma(\bar{n})).$$

Since $n_i^{N_{id}} = n_i$, and $n_i^{M_\sigma} = \sigma(n_i)$ for all i, by Claim 1.1,

$$N \models \varphi(\bar{n}) \text{ (in } L) \iff N_{id} \models \varphi((\bar{n})) \text{ (in } L \cup N\text{), and}$$

$$M \models \varphi(\sigma(\bar{n})) \text{ (in } L) \iff M_\sigma \models \varphi((\bar{n})) \text{ (in } L \cup N).$$

It follows that $\sigma : N \longrightarrow M$ is an elementary embedding if and only if N_{id} and M_σ have the same $L \cup N$-theory, namely $\Delta_e(N)$. □

Claim 1.3 If M and N are elementarily equivalent (with disjoint domains).

1. $\Delta_e(N) \cup \Delta_e(M)$ is a satisfiable $L \cup M \cup N$-theory.

2. There is an L-structure K in which both M and N embed elementarily.

Proof. 1. Let

$$\Sigma_0 = \{\varphi_1((\bar{n})), \dots, \varphi_k((\bar{n})), \phi_1((\bar{m})), \dots, \phi_\ell((\bar{m}))\}$$

be a finite subsets of $\Delta_e(N) \cup \Delta_e(M)$ with $\bar{n} \in N$ and $\bar{m} \in M$. Put

$$\varphi(\bar{x}) = \bigwedge_{i=1}^{k} \varphi_k(\bar{x}) \qquad \text{and} \qquad \phi(\bar{y}) = \bigwedge_{i=1}^{k} \phi_\ell(\bar{y}).$$

So $N \models \varphi(\bar{n})$ and $M \models \phi(\bar{m})$, hence $M \models \exists \bar{y}\phi(\bar{y})$. As $\exists \bar{y}\phi(\bar{y})$ is an L-formula, and as M and N have the same L-theory, one has $N \models \exists \bar{y}\phi(\bar{y})$, so there is a tuple $\bar{\alpha}$ in N such that $N \models \phi(\bar{\alpha})$. Interpreting n_i by $\bar{n}$, m_i by α_i (this is possible since N and M are disjoint) and any other constant symbol of $N \cup M$ arbitrarily, N is $L \cup M \cup N$-structure that is a model of Σ_0. By the Compactness Theorem, $\Delta_e(N) \cup \Delta_e(M)$ has a model.

2. Let $(K, L^K \cup N^K \cup M^K)$ be a model of $\Delta_e(N) \cup \Delta_e(M)$. Define the maps

$$\sigma : N \longrightarrow K,\ n \mapsto n^K \qquad \text{and} \qquad \tau : M \longrightarrow K,\ m \mapsto m^K.$$

The reduct $K_\sigma = (K, L^K \cup N^K)$ is a model of $\Delta_e(N)$, and $K_\tau = (K, L^K \cup M^K)$ is a model of $\Delta_e(M)$. By Claim 1.2, σ and τ are elementary L-embeddings. □

Exercise 2

Claim 2.1

Proof. □

Exercise 3

Claim 3.1

Proof. □

Exercise 4

Claim 4.1 1. An L_{ring}-formula $\varphi(\bar{x})$ is of the form $t(\bar{x}) = s(\bar{x})$ for some L_{ring}-terms s, t.

2. For any L_{ring}-term $t(\bar{x})$ with $\bar{x} = (x_1, \dots, x_n)$, there is a polynomial $P(\bar{x})$ in n variables with coefficients in $\mathbf{Z}$ such that, for every field K and $\bar{k} \in K$,

$$t^K(\bar{k}) = P(\bar{k}).$$

(note that if $P(\bar{x}) = a_{\alpha_m}\bar{x}^{\alpha_m} + \cdots + a_{\alpha_1}\bar{x}^{\alpha_1} + a_{\bar{0}}$ where $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in}) \in \mathbf{N}^n$, $a_i \in \mathbf{N}$ and $\bar{x}^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, then by definition, $P(\bar{k}) = a_{\alpha_m}\bar{k}^{\alpha_m} + \cdots + a_{\alpha_1}\bar{k}^{\alpha_1} + a_{\bar{0}}1_K$, so $P(\bar{k})$ is well-defined even if K has characteristic p.)

3. In particular, an L_{ring}-formula $\varphi(\bar{x})$ is equivalent modulo the theory of fields to $P(\bar{x}) = 0$ for some polynomial P with coefficients in $\mathbf{Z}$.

Proof. 1. By definition of an atomic formula, as $=$ is the only relation symbol in the language.

2. Similar to Exercise 1 in Sheet 2, where this was done for the particular case of $\mathbf{R}$. By induction on $c(t)$: if t is a constant symbol $c \in \{0, 1\}$, then $P(\bar{x}) = c$. If t is a variable x_i, then $P(\bar{x}) = x_i$. If t is of the form $t_1 * t_2$ with $* \in \{+, \times, -\}$, $t_1^K = P_1$ and $t_2^K = P_2$ then $t^K = P_1 * P_2$ hence $P = P_1 * P_2$, which is in $\mathbf{Z}[\bar{x}]$ since $\mathbf{Z}[\bar{x}]$ is a ring.

3. For any field K, any $\bar{k} \in K^n$ and P, Q in $\mathbf{Z}[\bar{x}]$, one has $P(\bar{k}) = Q(\bar{k}) \iff (P - Q)(\bar{k}) = 0$. □

Claim 4.2 Let M and N be two algebraically closed fields, $\bar{a}$ in M and $\bar{b}$ in N two n-tuples such that for any atomic L_{ring}-formula $\varphi(\bar{x})$, one has

$$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{b}). \tag{2}$$

Then (2) holds for any L_{ring}-formula $\varphi(\bar{x})$.

Proof. By induction on the complexity of φ. It holds for atomic formulas by assumption, and obviously holds for $\neg\varphi$ and $\varphi \wedge \psi$ as soon as it holds for φ and ψ. Assume that $M \models \exists y \psi(y, \bar{a})$. Then there is α in M such that $M \models \psi(\alpha, \bar{a})$. Let K be the prime field of M (i.e. $\mathbf{Q}$ or $\mathbf{Z}/p\mathbf{Z}$), and $K(\bar{a})$ the subfield of M generated by $\bar{a}$. Note that K is also the prime field of N by (2). **If α is algebraic over** $K(\bar{a})$, let $P_{\bar{a}}$ be its minimal polynomial:

$$P_{\bar{a}}(x) = \frac{S_0(\bar{a}) + S_1(\bar{a})x + \cdots + S_n(\bar{a})x^n}{S_n(\bar{a})} \qquad \text{for} \qquad S_0, \ldots, S_n \in \mathbf{Z}[\bar{x}].$$

Let $P_{\bar{b}}$ be the polynomial $\dfrac{S_0(\bar{b}) + S_1(\bar{b})x + \cdots + S_n(\bar{b})x^n}{S_n(\bar{b})}$ with coefficients $S_i(\bar{b})$ in N obtained replacing $\bar{a}$ by $\bar{b}$ (note that $P_{\bar{b}}$ is uniquely defined: if $S(\bar{a}) = T(\bar{a})$ for some S, T in $K(\bar{x})$, then $S(\bar{b}) = T(\bar{b})$ holds by (2)). As N is algebraically closed, there is a root β of $P_{\bar{b}}$ in N. If $Q(\alpha, \bar{a}) = 0$ for some polynomial $Q \in \mathbf{Z}[y, \bar{x}]$, then α is a root of the one variable polynomial $Q(y, \bar{a})$, so $Q(y, \bar{a}) = R_{\bar{a}} \cdot P_{\bar{a}}$ for some one variable polynomial $R_{\bar{a}}(y)$ with coefficients in $K(\bar{a})$. One has

$$R_{\bar{a}}(y) = \frac{T_0(\bar{a}) + T_1(\bar{a})y + \cdots + T_m(\bar{a})y^m}{T(\bar{a})} \qquad \text{for some } T, T_0, \ldots, T_m \in \mathbf{Z}[\bar{x}], \text{ and}$$

$$Q(y, \bar{a}) = Q_0(\bar{a}) + Q_1(\bar{a})y + \cdots + Q_{m+m}(\bar{a})y^m \qquad \text{for some } Q_0, \ldots, Q_{n+m} \in \mathbf{Z}[\bar{x}].$$

As equality of two polynomials is given by equality of their coefficients, one has

$$Q(y, \bar{a}) = R_{\bar{a}} \cdot P_{\bar{a}} \iff Q_i(\bar{a}) \cdot T(\bar{a}) \cdot S_n(\bar{a}) = \sum\nolimits_{p+q=i} T_p(\bar{a}) Q_q(\bar{a}) \text{ for all } 0 \leqslant i \leqslant n + m.$$

By (2), one must also have

$$Q_i(\bar{b}) \cdot T(\bar{b}) \cdot S_n(\bar{b}) = \sum\nolimits_{p+q=i} T_p(\bar{b}) Q_q(\bar{b}) \text{ for all } 0 \leqslant i \leqslant n + m,$$

hence the decomposition $Q(y, \bar{b}) = R_{\bar{b}} \cdot P_{\bar{b}}$ in $K(\bar{b})(y)$, so that we have $Q(\beta, \bar{b}) = 0$. By a symmetry argument, this shows that $(\alpha, \bar{a})$ and $(\beta, \bar{b})$ are roots of the same polynomials with coefficients in $\mathbf{Z}$, hence satisfy the same atomic formulas by Claim 4.1. By induction hypothesis, one has $M \models \psi(\alpha, \bar{a}) \iff N \models \psi(\beta, \bar{b})$. This shows that $N \models \exists \psi(y, \bar{b})$. The converse implication holds by symmetry of the assumptions. **If α is transcendental over** $K(\bar{a})$, let N_1 be an uncountable elementary extension of N. As the subsets of N_1 of elements that are algebraic over $K(\bar{b})$ is countable, there exists an element β in N_1 that is transcendental over $K(\bar{b})$. If $Q(\alpha, \bar{a}) = 0$ for some polynomial Q with coefficients in $\mathbf{Z}$, then $Q(x, \bar{a})$ must be the zero polynomial, hence $Q(x, \bar{b})$ is also zero by (2), so $Q(\beta, \bar{b}) = 0$. By a symmetric argument, this shows that $(\alpha, \bar{a})$ and $(\beta, \bar{b})$ are roots of the same polynomials with coefficients in $\mathbf{Z}$, hence satisfy the same atomic formulas by Claim 4.1. By induction hypothesis (applied to M and N_1), one has $N_1 \models \psi(\beta, \bar{b})$, so $N_1 \models \exists y \psi(y, \bar{b})$ hence $N \models \exists y \psi(y, \bar{b})$ (as $N \preceq N_1$). □

Claim 4.3 ACF is model complete.

Proof. Let N, M be two algebraically closed field with $N \subset_{L_{ring}} M$ (i.e. N is a subring of M). Let $\bar{a}$ be a tuple in N. For every atomic L_{ring}-formula $\psi(\bar{x})$, one has

$$N \models \psi(\bar{a}) \iff M \models \psi(\bar{a}).$$

By Claim 4.2, one also has $N \models \varphi(\bar{a}) \iff M \models \varphi(\bar{a})$ for every L_{ring}-formula $\varphi(\bar{x})$ and so $N \preceq M$. □