Model theory
8. Model companions

Exercise 1 (on torsion-free Abelian groups and divisible torsion-free Abelian groups) Let L_{mon} be the language $\{+, 0\}$ of monoids, Σ_1 the L_{mon}-theory of torsion-free Abelian groups and Σ_2 the L_{mon}-theory of divisible torsion-free Abelian groups. The aim of the exercise is to show that Σ_2 is a model-companion of Σ_1.

1. Determine the atomic L_{mon}-formulas and their interpretations in a model of Σ_1.
2. Let G, H be two models of Σ_2. Let $\bar{a}$ in G and $\bar{b}$ in H be two n-tuples such that for any atomic L_{mon}-formula $\varphi(\bar{x})$, one has
$$G \models \varphi(\bar{a}) \iff H \models \varphi(\bar{b}). \tag{1}$$
Show that (1) holds for any L_{mon}-formula $\varphi(\bar{x})$.
3. Show that every torsion-free Abelian group G embeds into a divisible torsion-free Abelian group.
4. Show that Σ_2 is the model companion of Σ_1.

Exercise 2 (On real fields and real-closed fields) A field K is called *real* if -1 cannot be written as a sum of squares of elements of K. A field K is called *real-closed* if it is real and has no proper real algebraic extension. Let L_{ring} be the language of rings, RF the L_{ring}-theory of real fields and RCF the L_{ring}-theory of real-closed fields. The aim of the exercise is to show that RCF is a model companion of RF. To this aim, we show that a real-closed field K has a canonical field ordering $\leqslant$, so that K has a natural $L_{ring} \cup \{<\}$-structure, and first prove that the $L_{ring} \cup \{<\}$-theory of real-closed field is model complete.

1. Show that every real field has a real-closed field extension.
2. If K is a real-closed field, show that every sum of squares in K is a square.
3. If K is a real-closed field, show that for every a in K, either a or $-a$ is a square.
4. Deduce that if K is a real-closed field, the relation
$$x \leqslant y \iff y - x \text{ is a square}$$
makes K into an ordered field. Show that $\leqslant$ is the unique such ordering on K.
5. Let K be a real-closed field. The aim of this question is to show that every irreducible polynomial over K has degree 1 or 2. For that, we consider i a root of $x^2 + 1$ in an extension of K and show that $K(i)$ is algebraically closed (the proof of this question can be skipped, and the result can be used in proving the next question).
   (a) Show that K has no algebraic extension of odd degree (by induction on the degree).
   (b) Show that every element of $K(i)$ has a square root in $K(i)$ and deduce that every polynomial of degree 2 with coefficients in $K(i)$ has a root in $K(i)$.

(c) Show that $K(i)$ has no algebraic extension. (hint: a finite group of exponent 2 has a normal subgroup of index 2)

(d) Deduce that every irreducible polynomial over K has degree 1 or 2.

6. (Intermediate Value Theorem) Let K be a real-closed field, $P \in K[x]$ and $a < b$ in K such that $P(a)P(b) < 0$. Show that there is some $c \in K$ such that $a < c < b$ and $P(c) = 0$.

7. Show that a quantifier-free $L_{ring} \cup \{<\}$-formula is logically equivalent modulo the $L_{ring} \cup \{<\}$-theory of real-closed fields to a disjunction of finitely many formulas of the form

$$\left(\bigwedge_{i=1}^{k} P_i(\bar{x}) > 0 \right) \wedge \left(\bigwedge_{i=1}^{\ell} Q_i(\bar{x}) = 0 \right)$$

for polynomials $P_1, \ldots, P_k, Q_1, \ldots, Q_\ell$ over $\mathbf{Q}$.

8. Let $F \subset K$ be two real-closed fields with their natural $L_{ring} \cup \{<\}$-structure. Show that F is existentially closed in K.

9. Deduce that RCF is model-complete (in L_{ring}), and that RCF is a model companion of RF.