Model theory
9. Axiomatisable classes

Exercise 1 (universal and existential axiomatisations) Let $\mathcal{C}$ be an axiomatisable class of L-structures.

1. Recall an equivalent condition to $\mathcal{C}$ being universally axiomatisable, and give examples and counterexamples of such classes $\mathcal{C}$.

2. A group G is called *locally finite* if every finitely generated subgroup of G is finite. Is a subgroup of a locally finite subgroup G locally finite? Is the class of locally finite subgroups universally axiomatisable in L_{gp}?

3. Show that if a class of L-structures $\mathcal{C}$ is existentially axiomatisable, then for all N in $\mathcal{C}$ and $N \subset_L M$, one has $M \in \mathcal{C}$.

Exercise 2 (a characterisation of axiomatisable classes) Let $\mathcal{C}$ be a class of L-structures. We say that $\mathcal{C}$ is closed under elementary equivalence if for every M in $\mathcal{C}$, every L-structure N that is elementary equivalent to M is in $\mathcal{C}$. We say that $\mathcal{C}$ is closed under ultraproducts if for every set I, every ultrafilter $\mathcal{U}$ on I and every family $(M_i)_{i\in I}$ of elements of $\mathcal{C}$, the ultraproduct $\prod_{\mathcal{U}} M_i$ is in $\mathcal{C}$.

1. Show that if $\mathcal{C}$ is axiomatisable, then $\mathcal{C}$ is closed under elementary equivalence and ultraproducts.

2. Show that if $\mathcal{C}$ is closed under elementary equivalence and ultraproducts, then $\mathcal{C}$ is axiomatisable.

3. Show that $\mathcal{C}$ is contained in a class of L-structures $\mathcal{D}$ that is axiomatisable.

4. Let $\mathcal{D}$ be the class of all L-structures M such that M is elementary equivalent to an ultraproduct of members of $\mathcal{C}$. Show that $\mathcal{D}$ is axiomatisable, and that it is the least axiomatisable class that contains $\mathcal{C}$.

Exercise 3 (on simple groups, again) In Exercise sheet 7, it was shown that if G is a simple group in the language L_{gp}, then every elementary substructure of G is also a simple group. Is the class of simple groups axiomatisable?